Calculus for Elementary & Middle School Teachers
Computing Derivatives & Integrals
Part 2

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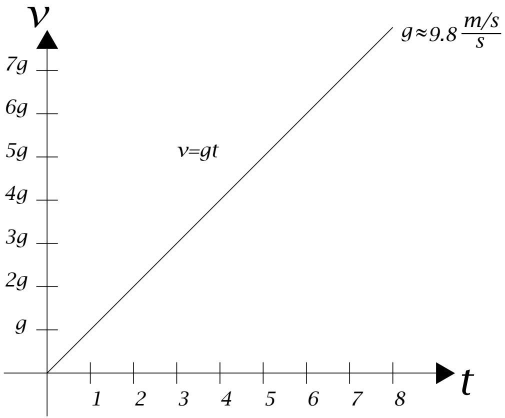
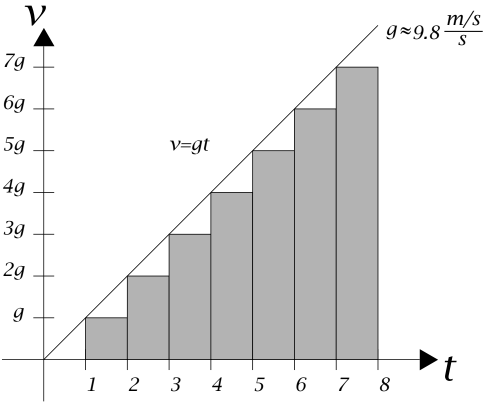
October 23, 2016

Distance Fallen is the Area Under the Velocity Curve



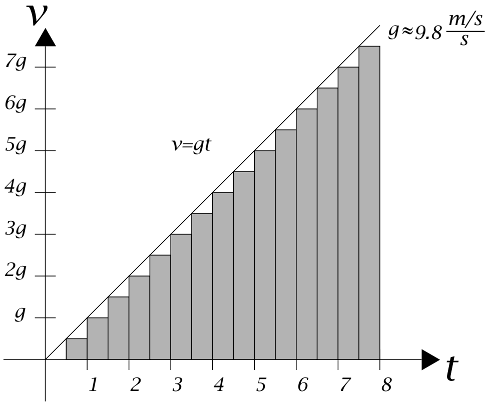
The graph of velocity as a function of time.

Distance Fallen is the Area Under the Velocity Curve



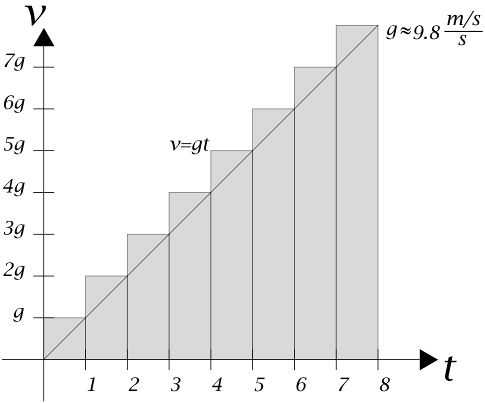
Estimate of distance traveled for $t = 8$ & $\Delta t = 1$, using the velocity at the beginning of each interval.

Distance Fallen is the Area Under the Velocity Curve



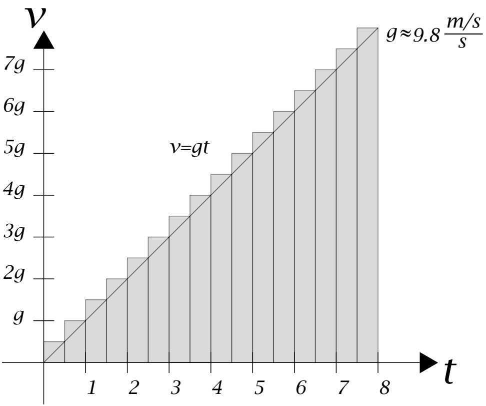
Estimate of distance traveled for $t = 8$ & $\Delta t = .5$, using the velocity at the beginning of each interval.

Distance Fallen is the Area Under the Velocity Curve



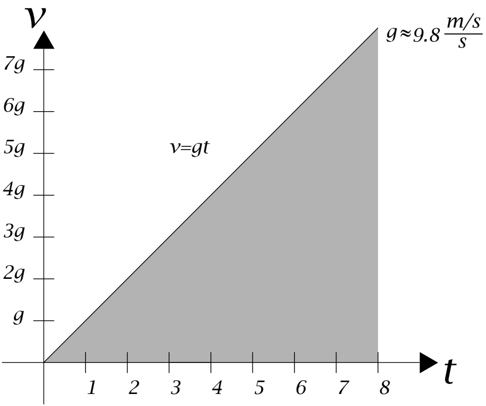
Estimate of distance traveled for $t = 8$ & $\Delta t = 1$, using the velocity at the end of each interval.

Distance Fallen is the Area Under the Velocity Curve



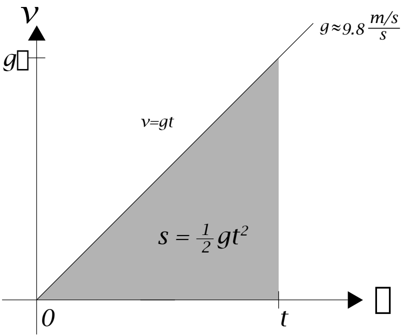
Estimate of distance traveled for $t = 8$ & $\Delta t = .5$, using the velocity at the end of each interval.

Distance Fallen is the Area Under the Velocity Curve



As $\Delta t \to 0$, these estimates converge to the *area* under the graph of $v = gt$ from $t = 0$ to $t = 8$.

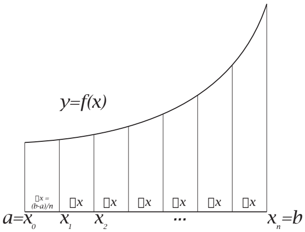
Distance Fallen is the Area Under the Velocity Curve



In general, as $\Delta t \to 0$, the distance fallen between times 0 and t is the area under the graph of $v = g\tau$ from $\tau = 0$ to $\tau = t$. We can derive the general formula from the area formula for a triangle with base t and height gt: $s = \frac{1}{2}gt^2$.

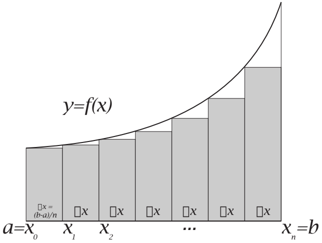
The Area Bounded by the Graph of a Function

- Now consider any function that is *continuous* on an interval, along with two points a and b in that interval.
- Let us focus first on the case that $a < b$ and f is positive and increasing on $[a, b]$.
- For each positive integer n, consider the *partition* of $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. (Δx clearly depends on n, but it is cumbersome to incorporate n into the notation.)

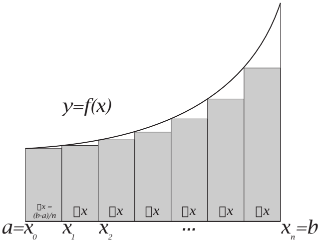


The Region Bounded by the n^{th} Lower Piece-wise Constant Approximation

- On each interval $[x_i, x_{i+1}]$, consider the minimum value of f; in this case, it will be $f(x_i)$.
- The area bounded by the graph of the constant function $g_i(x) = f(x_i)$ is $f(x_i)\Delta x$.
- The region bounded from a to b by the graph of the piecewise constant function $g(x) = g_i(x)$ for $x \in [x_i, x_{i+1}]$ is contained in the region bounded by the graph of f.



The Area bounded by the n^{th} Lower Piece-wise Constant Approximation



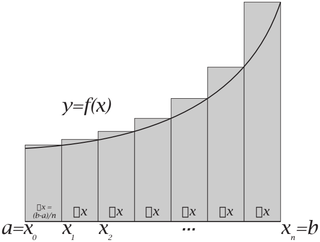
Its area is $L_n =$

$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

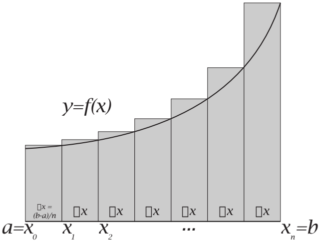
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

The Region Bounded by the n^{th} Upper Piece-wise Constant Approximation

- On each interval $[x_i, x_{i+1}]$, consider the maximum value of f; in this case, it will be $f(x_{i+1})$.
- The area bounded by the graph of the constant function $h_i(x) = f(x_{i+1})$ is $f(x_{i+1})\Delta x$.
- The region bounded from a to b by the graph of the piecewise constant function $h(x) = h_i(x)$ for $x \in [x_i, x_{i+1}]$ clearly contains the region bounded by the graph of f.



The Area Bounded by the n^{th} Upper Piece-wise Constant Approximation



Its area is $U_n =$

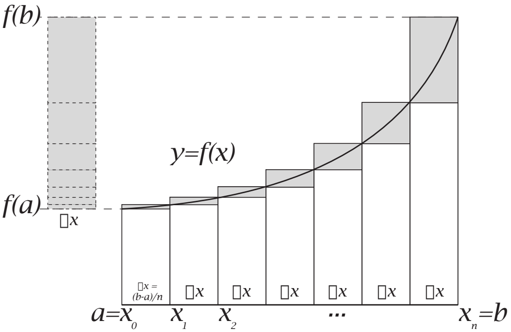
$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x.$$

The Limits as $n \to \infty$

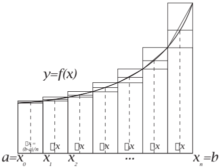
- As we let $n \to \infty$, L_n increases and U_n decreases. (Why?)
- Clearly $L_m \leq U_n$ for any m and n.
- By the continuity of the real number system, $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} U_n$ must exist.
- In fact, these limits are the same, as we will see in a moment.
- Since the region bounded by the function f contains the region bounded by each lower piece-wise constant approximation and is contained in the region bounded by each upper piece-wise constant approximation, its area must be this common limit.

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n =$ The Area of the Region bounded by f



- The difference between L_n and U_n, in this case, is $(f(b) - f(a))\Delta x_n$.
- As $n \to \infty$, $\Delta x_n \to 0$; hence this difference goes to 0.

Other Approximations



- Furthermore, all other approximations, such as the midpoint approximation, are squeezed between L_n and U_n as well.
- These other approximations converge more quickly than L_n and U_n.
- But we can often calculate the area exactly in another way.
- To do this, we need to look at the rate at which area varies, and find a function that varies at that rate.

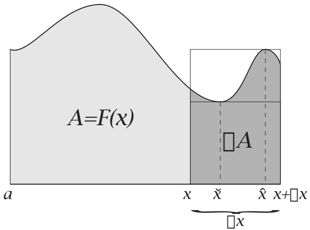
Area as a Variable Quantity

- Consider the area bounded by the graph of a continuous function f from an initial input a to x.
- This area is a function of x: call it $A = F(x)$.
- We will first compute $F'(x)$ and use it to compute $F(x)$.
- Note that to do this we must view the area bounded by f as a quantity that *varies* with the ending point of the interval, x, just as we viewed distance as a quantity that varied with time.
- Recall that

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}.$$

Bounding the Derivative of the Area

- Let $f(\check{x})$ be the minimum value of f on the interval form x to $x + \Delta x$, and let $f(\hat{x})$ be the maximum value of f on the interval form x to $x + \Delta x$.



- Then $f(\check{x})\Delta x \leq \Delta A \leq f(\hat{x})\Delta x$; hence,

$$f(\check{x}) \leq \frac{\Delta A}{\Delta x} \leq f(\hat{x}).$$

Computing the Derivative of the Area Using the Squeeze Theorem

- Since $\check{x}$ and $\hat{x}$ are between x and $x + \Delta x$, $\check{x} \to x$ and $\hat{x} \to x$ as $\Delta x \to 0$.
- Since f is continuous, $\lim_{\check{x} \to x} f(\check{x}) = \lim_{\hat{x} \to x} f(\hat{x}) = f(x)$.
- Combining the two previous observations, we obtain

$$\lim_{\Delta x \to 0} f(\check{x}) = \lim_{\Delta x \to 0} f(\hat{x}) = f(x).$$

- Thus, by the Squeeze Theorem,

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x)!$$

- The fact that $F' = f$ and the initial condition $F(a) = 0$ completely determine the function F. Any function whose derivative is f will compute the area as long as you subtract its value at a so it agrees with F.

The Fundamental Theorem of Calculus

To summarize, if you want to compute the area under the graph $y = f(x)$ from $x = a$ to $x = b$:

- Find *any* function G such that $G' = f$.
- The desired area is $G(b) - G(a)$.
- That's it! This powerful result is called the *Fundamental Theorem of Calculus*.
- Let's use our knowledge of derivatives to find some areas bounded by curves!