# MAT 4810: Calculus for Elementary and Middle School Teachers

# Computing Derivatives & Integrals Part I

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# Believe You Can Solve Problems Without Being Told How!



I would like to share the following words from the introduction to Don Cohen's book. I hope you will take the attitude they express to heart and share it with your own students.

"You can do it! You must tell yourself that. Don't think because you haven't done a problem before, that you can't do it, or that someone else must show you how to do it first (a myth some people want to use to keep others in ignorance). You can do it! Don't be afraid. You've learned how to do the hardest things you'll ever do, walking and talking – mostly by yourself. This stuff is much easier!"

# Recall: Linear Rates of Change

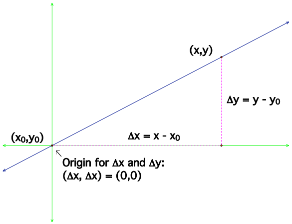


- To travel at 60 miles per hour ($\frac{\text{mi}}{h}$) is to travel a distance of $60\Delta t$ miles in $\Delta t$ hours: $\Delta d = 60\Delta t$, where $d$ is distance in miles and $t$ is time in hours; $\Delta d$ and $\Delta t$ are the changes in these quantities, respectively.
- A *rate* such as miles per hour – that is, a ratio of the change in output over the change in input of a function – gives a *purely linear* relationship between these changes.
- That is, the change in output is simply a multiple of the change in input.

# Recall: Non-Linear Rates of Change

- For most functions, the relationship between a change in input and the resulting change in output is *not* linear.
- This is because the ratio between these changes is not constant: the change in output is not a constant multiple of the change in input.
- When you go on a road trip, you don't drive at the same velocity all the time. Your average velocity over different intervals of time will vary.
- Only those functions whose graphs are straight lines exhibit a constant linear relationship between any change in input and the resulting change in output. The graphs of most functions are are curved.

# The Equation of a Line




- Given a fixed real number $m$, some initial input value $x_0$, and corresponding output value $y_0 = f(x_0)$:

$$\frac{\Delta y}{\Delta x} = m \Leftrightarrow \frac{y - y_0}{x - x_0} = m \Leftrightarrow y - y_0 = m(x - x_0).$$

- Constant rate of change is the defining property of those functions whose graphs are straight lines.

# Linear Functions are Central – But Not Enough



- Linear functions are relatively simple to understand and have many useful properties.
- There is a whole field of mathematics called *linear algebra*.
- But, as even the example of a road trip illustrates, linear functions alone are not adequate for modeling reality or fully developing the capacity of mathematics and science.
- To take advantage of the properties of linear functions, we *study* more complicated functions *using* linear functions.
- We do this by studying the linear relationship between changes in input and output *at each point* where such a relationship applies.

# Recall: The Definition of the Derivative Function



- Let $y = f(x)$ define a function $f$.
- The definition of the derivative function, $f'$, is

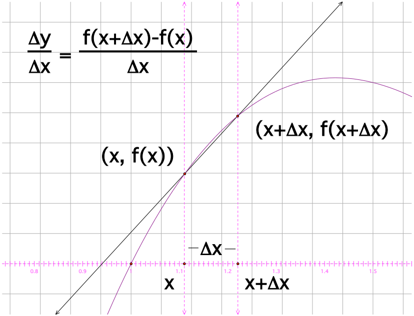
$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

*if this limit exists.*

- $\frac{\Delta y}{\Delta x}$ will not necessarily approach a limit as $\Delta x \to 0$. If $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ exists, $f$ is *differentiable* at $x$.
- This definition is based on the idea that $f'(x)$ represents the instantaneous rate of change of $f$ at input $x$.
- In other words, $f'(x)$ gives the linear relationship between changes in input and output *at point* $(x, y)$.
- In general, this rate will vary from point to point.
- Let's look at this graphically.

# The Slope of a Curve

- The value of the derivative $f'(x)$ gives the slope of the curve $y = f(x)$ at the point $(x, y)$.




- If $f'(x)$ is defined, the curve is "smooth" at $(x, f(x))$.
- Under increasing magnification, it would look straighter and straighter.

# Examples & a Pattern

- Using this definition, we were able to calculate the derivatives of the following functions:
  - If $f(x) = c$, where $c$ is a constant number, then $f'(x) = 0$.
  - The derivative of the identity function $i$, defined by $i(x) = x$, is given by $i'(x) = 1$.
  - If $f(x) = x^2$, then $f'(x) = 2x$.
  - If $f(x) = x^3$, then $f'(x) = 3x^2$.
  - If $f(x) = x^{-1}$, then $f'(x) = -x^{-2}$.
- A clear pattern emerges, leading us to the following conjecture:

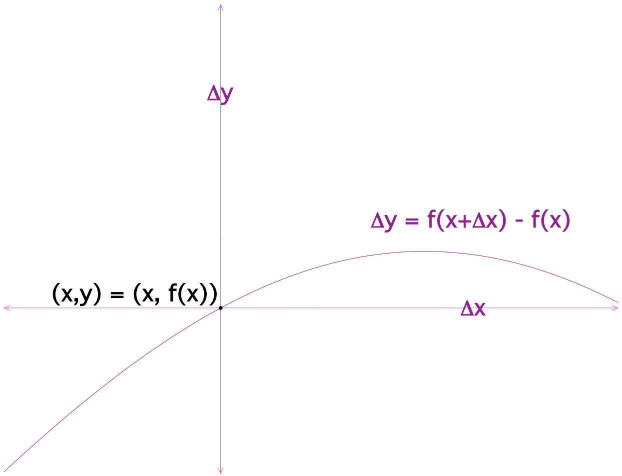
  If $f(x) = x^n$, where $n$ is an integer, then $f'(x) = nx^{n-1}$.
- We will verify this conjecture shortly, but first let's look graphically at these functions and their derivatives.

# Local Coordinates: The $(\Delta x, \Delta y)$-Coordinate System

- Since the slope of the curve $y = f(x)$ generally varies from point to point, it is helpful to view the curve in *local coordinates* based at each point $(x, y) = (x, f(x))$.
- By local coordinates, we mean that we place the origin at $(x, y) = (x, f(x))$.
- We have already considered the $(\Delta x, \Delta y)$-coordinate system, in which we graph $\Delta y = f(x + \Delta x) - f(x)$. (Note that $x$ is fixed in this equation.)
- Thus, $\Delta x$ denotes change in input away from the value $x$ and $\Delta y$ denotes the corresponding change in the output of $f$ away from the value $y = f(x)$.
- The graph $\Delta y = f(x + \Delta x) - f(x)$ is just a *translation* of the graph of $f$ taking a fixed value $(x, y) = (x, f(x))$ to the origin.

# The Graph $\Delta y = f(x + \Delta x) - f(x)$



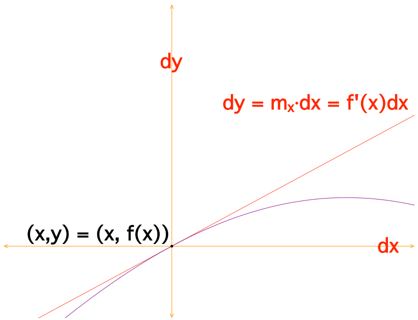


# Local *Differential* Coordinates: The $(dx, dy)$-Coordinate System



- When we want to look at the slope of the curve $y = f(x)$ at a particular point $(x, y) = (x, f(x))$, it is helpful to graph the *tangent line* to the curve at $(x, y)$.
- The tangent line is simply the straight line passing through $(x, y)$ with slope $f'(x)$.
- Thus, the slope of the tangent line matches the slope of the curve at $(x, y)$.
- We use local coordinates $(dx, dy)$ to graph the tangent line function, which is defined by $dy = f'(x)dx$. (Note that $x$, and hence $f'(x)$, is fixed in this equation.)

# The Graph $dy = m_x \cdot dx = f'(x)dx$




- Thus, $dx$ denotes a change in input away from the value $x$ and $dy$ denotes the corresponding *linear* change in output that applies at that point.
- $dy$ is called the *differential* change in $f$ resulting from a *differential* change in input $dx$ from $x$.

# Recall: Algebraic Combinations of Functions

- We create new functions by combining functions algebraically. For example, if $f(x) = x^2$, $g(x) = 5$, and $i(x) = x$:
  - We can define the function $f + i$ by the definition $(f + i)(x) = f(x) + i(x) = x^2 + x$.
  - We can define the function $gi = 5i$ by the definition $gi(x) = 5i(x) = 5x$.
  - We can define the function $gf = 5f$ by the definition $gf(x) = 5f(x) = 5x^2$.
  - We can define the function $5fi$ by the definition $(5fi)(x) = 5(x^2)(x) = 5x^3$.
  - We can define the function $f + gi = f + 5i$ by the definition $(f + 5i)(x) = f(x) + 5i(x) = x^2 + 5x$.
  - We can define the function $\frac{1}{f}$ by the definition $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)} = \frac{1}{x^2} = x^{-2}$. Note that the domain of $\frac{1}{f}$ does not include 0.

# Complicated Derivatives from Simpler Ones



- Rules have been developed for computing derivatives of algebraic combinations of functions.
- The purpose of these rules is to compute the derivatives of functions using the derivatives of simpler ones.
- Our emphasis will be on understanding why the rules work. Therefore, we will restrict our attention to the sum rule and the product rule, which are sufficient to compute enough derivatives to explore some applications.
- For example, since we know the derivatives of the functions defined by $f(x) = x^2$, $g(x) = x^3$, and $h(x) = 5$, these rules will allow us to compute the derivative of the function defined by $j(x) = 5x^2 + x^3 + x^5$.

# The Rules for Derivatives are Not So Simple!

- **Warning!** The formula that defines $f'(x)$ is the limit of a somewhat complicated function of $\Delta x$ based on the values of $f$ at the two inputs $x$ and $x + \Delta x$:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

  Therefore, we cannot expect the derivatives of algebraic combinations to depend on the derivatives of the functions being combined in a naively simple manner – and they don't!
- In particular, the derivative of a product is *not* the product of the derivatives.

# Summary of Useful Notation

- Let $y = f(x)$ and $z = g(x)$. Then, in local coordinates:
- $\Delta y = f(x + \Delta x) - f(x)$ and $\Delta z = g(x + \Delta x) - g(x)$.
- Thus $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ and $g'(x) = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x}$; and
- Please remember that an expression such as $\Delta x$ is a *single* variable representing a *single* number. They two symbols (such as $\Delta$ and $x$) cannot be separated.
- Think of $\Delta x$ as "the change in $x$". Clearly, (the change in $x)^2$ is *not* (the change in$)^2$ times $x^2$!
- Similarly $\Delta y$ (the change in y), $dx$ (the differential change in $x$), and $dy$ (the differential change in $y$) are single variables that cannot be separated.
- *Some pictures will help us understand limits of sums, differences, quotients, and products.*

# The Derivative of a Sum is the Sum of the Derivatives



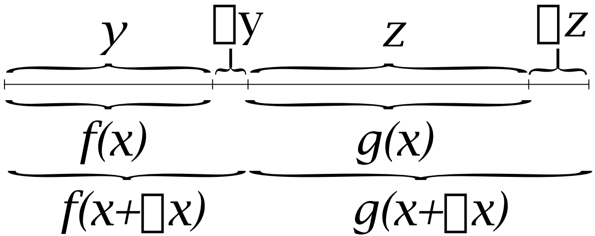
**Theorem**

*If $f$ and $g$ are differentiable at $x$, then so is $f+g$, and $[f+g]'(x) = f'(x) + g'(x) = [f'+g'](x)$.*

- This one is simple!

# Proof that $[f+g]' = f' + g'$






**Proof.**

$[f+g]'(x) = \lim_{\Delta x \to 0} \frac{\Delta[y+z]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y + \Delta z}{\Delta x} =$

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = f'(x) + g'(x)$ $\square$

# The Derivative of a Product is *Not* the Product of the Derivatives



**Theorem**

*If $f$ and $g$ are differentiable at $x$, then so is $fg$, and $[fg]'(x) = f(x)g'(x) + f'(x)g(x) = [fg' + f'g](x)$.*

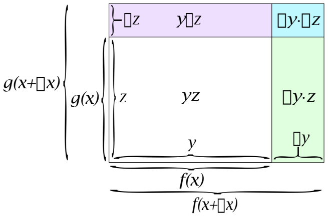
**Corollary**

*If $c \in \mathbb{R}$ and $g$ is differentiable at $x$, then $[cg]'(x) = cg'(x)$.*

**Proof of corollary.**

The derivative of a constant function is the constant function with output 0: if $f(x) = c$, then $f'(x) = 0$. Thus, applying the product rule, we obtain $[cg]'(x) = cg'(x) + 0 \cdot g(x) = cg'(x)$ □

# Proof that $[fg]' = fg' + f'g$




**Proof.**

$[fg]'(x) = \lim_{\Delta x \to 0} \frac{\Delta[yz]}{\Delta x} = \lim_{\Delta x \to 0} \frac{y\Delta z + \Delta y \cdot z + \Delta y \cdot \Delta z}{\Delta x} =$

$y \cdot \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} + \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot z + \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta z =$

$f(x)g'(x) + f'(x)g(x) + f'(x) \cdot 0 = [fg' + f'g](x)$ □

# Examples

- Let $h(x) = x^4$. We will verify that $h'(x) = 4x^3$.
  - Consider $f(x) = x^3$ and $g(x) = x$. $h(x) = f(x)g(x)$.
  - From the definition of the derivative, we have computed that $f'(x) = 3x^2$ and $g'(x) = 1$.
  - Therefore, by the product rule, $h'(x) = f(x)g'(x) + f'(x)g(x) = x^3 \cdot 1 + 3x^2 \cdot x = 4x^3$.
- Let $f(x) = 5x^4$. Compute $f'(x)$?
  - Let $g(x) = x^4$. Then $f = 5g$.
  - By the special case of the product rule in which one function is constant, $f'(x) = 5g'(x) = 5(4x^3) = 20x^3$.
- Let $f(x) = 5x^4 + x^3 + 2$. Compute $f'(x)$.
  - By the sum rule, $f'(x) = 20x^3 + 3x^2 + 0 = 20x^3 + 3x^2$.