MAT 3530: Algebra **Name:** ***Solutions***

Preliminary Exam: take-home portion.

1. (a) Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective, then f is injective.

   Proof. Assume $g \circ f$ is injective, and let $f(a) = f(a')$. Then $g \circ f(a) = g(f(a)) = g(f(a')) = g \circ f(a')$. Since $g \circ f$ is given to be injective, it follows that $a = a'$, by definition of injectivity; therefore, f is injective by definition. □

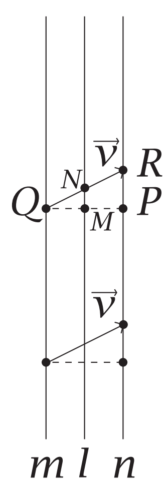
   (b) Is the converse true? If so, prove it; if not, give a counterexample.

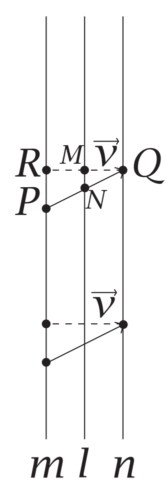
   The converse if false. There are many possible counterexamples. For example, the identity function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is certainly injective. Consider $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$. Then $g \circ f = g$ is certainly not injective: $g \circ f(1) = 1 = g \circ f(-1)$, $g \circ f(2) = 4 = g \circ f(-2)$, etc.

2. Let $\tau_{\vec{v}}$ denote translation by the vector $\vec{v}$, and let ϕ_l denote reflection in the line l.

   Proof. In either case we know the composition is orientation reversing; therefore it must be a glide reflection (in general) or a reflection (if the glide vector is $\vec{0}$). Thus, to identify the result of the composition, it suffices to find the line that is fixed and the vector by which this line glides along itself.

   (a) The general case that $\vec{v}$ and l are not perpendicular:

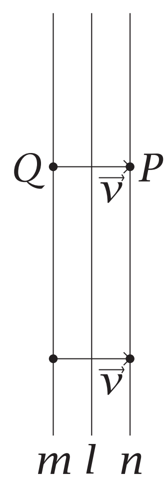




   Consider points P, Q, and R arranged so that $\vec{v}$ is the vector from Q to R, line l bisects QR, and $PQ \perp l$. Let m and n be the lines through Q and P, respectively, such that $m \parallel l \parallel n$. Since triangles $\triangle PQR$ and $\triangle MQN$ are right triangles sharing an angle, they are similar; hence M is the midpoint of of PQ. Therefore, $\tau_{\vec{v}} \circ \phi_l(P) = \tau_{\vec{v}}(Q) = R$. (See figure at left.) All other points of line n are similarly transformed to points on line n with glide vector $\overrightarrow{PR}$. In other words, $\tau_{\vec{v}} \circ \phi_l$ is a glide reflection in n with glide vector $\overrightarrow{PR}$.

Similarly, consider points P, Q, and R arranged so that $\vec{v}$ is the vector from P to Q, line l bisects PQ, and $QR \perp l$. Let m and n be the lines through P and Q, respectively, such that $m \parallel l \parallel n$. Since triangles $\triangle PQR$ and $\triangle MQN$ are right triangles sharing an angle, they are similar; hence M is the midpoint of of QR. Therefore, $\phi_l \circ \tau_{\vec{v}}(P) = \phi_l(Q) = R$. (See figure at right.) All other points of line m are similarly transformed to points on line m with glide vector $\overrightarrow{PR}$. In other words, $\phi_l \circ \tau_{\vec{v}}$ is a glide reflection in m with glide vector $\overrightarrow{PR}$.

(b) The special case that $\vec{v}$ and l are perpendicular.



Consider points P and Q arranged so that $\vec{v}$ is the vector from Q to P and line l bisects PQ. Then $\tau_{\vec{v}} \circ \phi_l(P) = \tau_{\vec{v}}(Q) = P$, and all other points on line n are similarly fixed. Thus $\tau_{\vec{v}} \circ \phi_l$ is simply a reflection in line n in this case. Similarly, $\phi_l \circ \tau_{\vec{v}}(Q) = \phi_l(P) = Q$, and all other points on line m are similarly fixed, so $\phi_l \circ \tau \vec{v}$ is a reflection in line m.

□