MAT 3530: Algebra **Name:** ______________________________

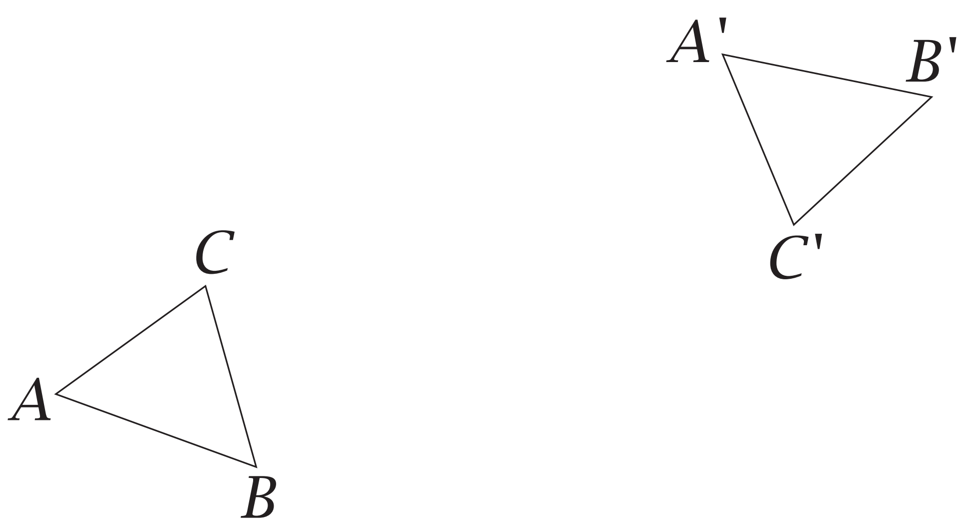
Midterm Exam: take-home portion.
Due by 5 p.m. on Monday, March 26. *Solutions must be typed in Latex.*

You are expected to work on this exam alone and to refrain from talking about the exam to anyone except the professor until the time and date when it is due. You may use your own notes and any published materials that you like. Published sources (whether hard-copy or on the Web) must be appropriately cited!

Your signature below attests to a pledge that you have done the exam according to the above instructions. (Please attach this cover page to your solutions.)

Signature:______________________________

1. Construct three lines l, m, and n such that the $\phi_n \circ \phi_m \circ \phi_l(A) = A'$, $\phi_n \circ \phi_m \circ \phi_l(B) = B'$, and $\phi_n \circ \phi_m \circ \phi_l(C) = C'$. (Remark: There are many ways to do it!)



2. Let l, m, and n be integers such that l divides mn and the greatest common divisor of l and m is k. Prove that l divides kn.

3. We proved that if p is a prime integer, m and n are integers, and p divides mn, then either p divides m or p divides n. In applications, we actually needed a more general result:

   Lemma. *If p is a prime integer, $m_1, m_2, m_3, \ldots, m_k$ are integers, and p divides $m_1 m_2 m_3 \cdots m_k$, then p divides m_i for some i.*

   Prove it!

4. Prove that for any natural number n, 3 divides $n^3 - n$.

5. Consider the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. (For example, the functions defined by $f(x) = |x|$, $g(x) = x^2$, and $h(x) = \cos x$ are in this set; the function defined by $j(x) = x^{-1}$ is not, since it is not defined, and certainly not continuous, at $x = 0$.) Just as we identify numbers with constant polynomials, we will more generally identify numbers with constant functions; for example, π denotes the function defined by $\pi(x) = \pi$ for all $x \in \mathbb{R}$ Using the same symbol for a number and constant function should cause no confusion. As should be familiar from precalculus, we can define the arithmetic operations of addition and multiplication of functions as follows, using the addition and multiplication of real numbers:

   - Given continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we define $f + g$ by $[f + g](x) = f(x) + g(x)$. (The brackets are unnecessary; I have used them to emphasize that $f + g$ denotes a single function whose output for x is $f(x) + g(x)$.)
   - Similarly, given continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we define fg by $[fg](x) = f(x)g(x)$. (Multiplication of functions in this sense is different from composition. We will always use the symbol "$\circ$" to denote composition.)

   In what follows you may use without proof or reference the results about continuous functions proven in Calculus 1. In particular, recall that constant functions and the identity function ($i(x) = x$) are continuous and that sums, products, and compositions of continuous functions are continuous. (It follows that polynomial functions are continuous.) In addition, quotients of continuous functions are continuous at the points at which they are defined. Thus, the function defined by $f(x) = \frac{\sin x}{x^2+1}$ is continuous, since $x^2 + 1$ is not 0 for any real value of x. (If we were considering complex valued functions, this quotient would not be continuous, but the function defined by $g(z) = \frac{\sin z}{e^z}$, for example, would.)

   (a) Prove that, with the operations of addition and multiplication of functions defined above, the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ constitutes a commutative ring. This ring is denoted $C^0(\mathbb{R})$. (The superscript is used because $C^1(\mathbb{R})$ denotes the ring of differentiable functions with continuous derivatives, $C^2(\mathbb{R})$ denotes the ring of twice differentiable functions with continuous second derivatives, and so on.)

   (b) Prove or disprove by giving a counter-example: $C^0(\mathbb{R})$ is an integral domain. (Hint: Consider piecewise-defined continuous functions.)

   (c) Describe the group of units of $C^0(\mathbb{R})$. (Hint: When is $\frac{1}{f}$ a continuous function on all of $\mathbb{R}$?)