MAT 3530: The Isometry Group of the Euclidean Plane, Isom($\mathbb{R}^2$)

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Due at the beginning of class on Friday, February 3.

1 Introduction

We have looked at several examples of groups and their geometric interpretations. By analyzing and computing in these examples, you will be better prepared to understand the theory of groups.

The isometry group of the Euclidean plane is, by definition, the group of all transformations of the plane that preserve distances. There are several ways to approach the Euclidean plane. The classical approach is through axioms about lines and points, as initiated by Euclid in about 300 B.C.E and completed after a long and rich history by David Hilbert in about 1900 A.C.E. We implicitly take this more abstract perspective when we do constructions with straightedge and compass. As is common among modern geometers and group theorists, we will take the simpler, constructive approach to defining the Euclidean plane, using its familiar Cartesian representation as the set of all ordered pairs of real numbers, $\mathbb{R}^2$, given a geometric structure by the distance function, or *metric*, imposed by the famous theorem of Pythagoras in axiomatic Euclidean geometry: $d\big((x_1, y_1), (x_2, y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry of the plane if $d\big(f(x_1, y_1), f(x_2, y_2)\big) = d\big((x_1, y_1), (x_2, y_2)\big)$ for all points (x_1, y_1) and (x_2, y_2). (An isometry of a geometric object need not be bijective; for example, the upper half-plane is isometrically mapped *into*, but not *onto*, itself by upward translation: $(x, y) \mapsto (x, y + a)$, where a is a positive constant. However, all isometries of $\mathbb{R}^2$ turn out to be bijective: as we have seen by our analysis in class, they are all compositions of up to three reflections and may be categorized as translations, rotations, reflections, and glide reflections, all of which are clearly invertible.)

The composition of two isometries is an isometry, the identity map $((x, y) \mapsto (x, y))$ is clearly an isometry, and all isometries of the plane are invertible; hence, the set of all isometries of $\mathbb{R}^2$ under the operation of composition satisfies the definition of a group. The usual name for this group is Isom($\mathbb{R}^2$). Isom($\mathbb{R}^2$) is an enormous, uncountably infinite, continuous group. It has many subgroups with a variety of properties and will provide us with a rich trove of examples as we study group theory, aside from being of great mathematical and scientific interest in its own right.

We will denote the different types of isometries as follows:

- Reflection in a line l will be denoted by ϕ_l. ("Phī" for "re*fl*ection"!)
- Rotation in *directed* angle θ with center O will be denoted by $\rho_{\theta,O}$. ("Rhō" for "*ro*tation"! For simplicity, we will consider all rotations to be directed counter-clockwise; this will also avoid the nuisance of having to put arrows in pictures. Clockwise rotation by α is equivalent to counter-clockwise rotation by $\theta = 2\pi - \alpha$.)
- Translation by the vector $\vec{v}$ will be denoted by $\tau_{\vec{v}}$. ("Tau" for "*t*ranslation"!)
- Glide reflection by the vector $\overrightarrow{PQ}$ (that is, translation by $\overrightarrow{PQ}$ composed with reflection in line $\overleftrightarrow{PQ}$ will be denoted by $\gamma_{\overrightarrow{PQ}}$. ("Gamma" for "*g*lide reflection"!)

2 Exercises

In what follows, *construct* means to do so with straightedge and compass (only). Arbitrary given points may be introduced (generally, I will provide them in my diagrams), but all other points must be determined by the intersection of a circle or line with another circle or line. Similarly, arbitrary given lines may be introduced, but all other lines must be determined by two already constructed points. No guesswork!

1. Prove that the composition of two injective functions is injective.

   [You can erase this and type your solutions here!]

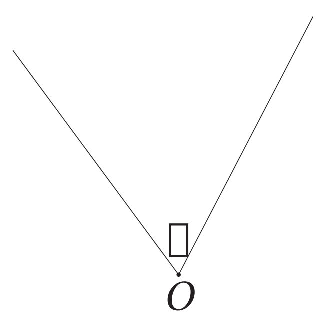
2. Prove that the composition of two surjective functions is surjective.

3. Prove as an immediate corollary of the two results above that the composition of two bijective functions is bijective.

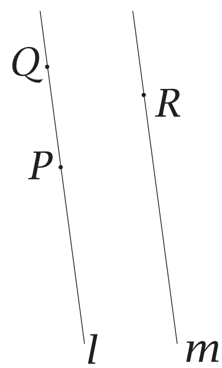
4. Prove formally that the composition of two isometries is an isometry.

5. We proved in class that the composition of reflections in distinct intersecting lines is a rotation by twice the angle between the lines, with center at the point of intersection.

   Conversely, any rotation may be described as the composition of two reflections through its center. Construct lines l and m such that $\rho_{\theta,O} = \phi_m \circ \phi_l$, with O and θ given below. You may either do the construction below by hand or substitute a PDF file made with a geometry application.

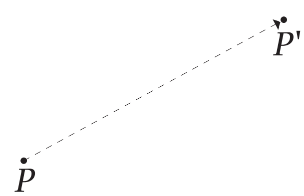


6. Given two parallel lines l and m, construct the images of given non-collinear points P, Q, and R under the composition $\phi_m \circ \phi_l$.

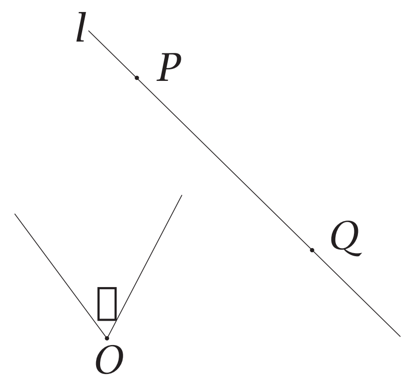


It should be clear from your construction that $\phi_m \circ \phi_l$ is a translation. Describe the length and direction of the vector of translation.

7. Conversely, any translation may be described as a composition of two reflections. Given translation vector $\vec{v} = \overrightarrow{PP'}$ below, construct lines l and m such that $\phi_m \circ \phi_l = \tau_{\vec{v}}$.



8. Given center O, angle of rotation θ, and directed line $\overleftrightarrow{PQ}$ (that is, oriented in the direction from P to Q), construct the images of P' and Q' under the rotation $\rho_{\theta,O}$. Construct directed line $\overleftrightarrow{P'Q'}$ and prove that the angle from directed line $\overleftrightarrow{PQ}$ to directed line $\overleftrightarrow{P'Q'}$ is θ. [Hint: Use similar triangles.]



9 & 10. Prove that the composition of two rotations, $\rho_{\theta,O}\circ\rho_{\theta',O'}$, is a rotation (general case) or a translation (special case). In the case that it is a rotation, identify its center and angle of rotation in terms of θ, θ', O, and O'. Using arbitrary representative centers and angles, construct the center of the composition $\rho_{\theta,O}\circ\rho_{\theta',O'}$. In the case that it is a translation, construct (in a separate diagram) the vector of translation. [Hint: Position the sides of the angles so that line $\overleftrightarrow{OO'}$ bisects them.]