MAT 3271: Selected Solutions to the Assignment 6

Chapter 2: Major Exercises

1. Since a projective plan is a model of incidence geometry, Incidence Axioms 1-3 and Propositions 2.1-2.5 (which follow logically from Incidence Axioms 1-3) hold for M. In addition, every line in M is incident with at least three points (Axiom I-2+), and any two lines intersect (Elliptic Parallel Property). We must show that the dual interpretation, M', satisfies the definition of a projective plane, using the known properties of M. To do this, *we must use the definitions of points and lines in interpretation M'.*

   Proposition: M' satisfies Incidence Axiom 1.

   This translates, using the definition of M', into the following statement: Given two lines of M, there is a unique point incident with both of them. Existence follows from the Elliptic Parallel Property; uniqueness follows from Proposition 2.1.

   Proposition: M' satsifies Incidence Axiom 2+; that is, every line is incident with at least three distinct points.

   This translates into: Every point of M is incident with at least three distinct lines. This follows immediately from the following lemma about projective planes. (A *lemma* is a theorem that is mainly used for proving another result.)

   Lemma. *In a projective plane, every point is incident with at least three distinct lines.*

   Proof. Let P be any point (of a given projective plane). There is a line l not incident with P, by Proposition 2.4. Line l is incident with at least three points, by Axiom I-2+; so let Q, R, and S be points on l. Consider lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$,and $\overleftrightarrow{PS}$, which exist by Axiom I-1.

   Claim: Lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$,and $\overleftrightarrow{PS}$ are distinct. Suppose, by way of contradiction, that $\overleftrightarrow{PQ} = \overleftrightarrow{PR}$. It follows (since all three points are collinear) that P is incident with line $\overleftrightarrow{QR} = l$, which contradicts our choice of l. A similar proof shows the other pairs of lines are distinct. □

   Proposition: There are three points of M' that are not collinear (that is, M' satisfies IA3).

   This translates as: There are three lines of M that are not concurrent. This is the statement of Proposition 2.2.

   Proposition: M' satisfies the Elliptic Parallel Property.

   This translates as: If P and Q are points of M, there is a line incident with both of them. This is the existence part of Incidence Axiom 1.

   The second part of this exercise was largely done in class. I did not grade it, since it didn't make sense to average two parts of a problem that had very little to do with each other.

2. As suggested by the hint, we will show that in an affine plane with a finite number of points, every line has the same number of points on it. Let l and m be distinct lines.

   Claim: there is a line distinct from l and m that intersects both of them. To prove the claim, first observe that there is a point P on l that does not lie on m and a point P' on m that does not lie on l. If l and m intersect, IA2 gives a point on each distinct from the point of intersection. If l and m are parallel, then any point on one is not on the other. Clearly line $\overleftrightarrow{PP'}$ is distinct from l and m and intersects both.

   Let Q be any point on l. If Q is distinct from P, then there is a unique line n through Q parallel to $\overleftrightarrow{PP'}$.

   Claim: n intersects m. If not, then m and $\overleftrightarrow{PP'}$ would be distinct lines through P' parallel to n. This contradicts the Euclidean Parallel Property. (This is how we use the definition of affine plane.)

   Let Q' be the point of intersection of n and m. (There is only one, by Propostion 2.1. You'll be surprised how often you use Proposition 2.1!) In this manner, we have defined a mapping $Q \mapsto Q'$ from the points on l to the points on m. (Of course, $P \mapsto P'$. Note also that if l and m intersect, the point of intersection maps to itself; it counts as a point on each line.)

Claim: This mapping is 1-1. First, if $Q \neq P$ then $Q' \neq P'$, since by construction $\overleftrightarrow{QQ'} \parallel \overleftrightarrow{PP'}$. So consider the case that $Q' = R'$, and neither Q nor R is equal to P. By the definition of our mapping, lines $\overleftrightarrow{QQ'}$ and $\overleftrightarrow{RR'}$ are both parallel to $\overleftrightarrow{PP'}$, and they both pass through $Q' = R'$. So by the Euclidean Parallel Property, they are the same line, and by Proposition 2.1 $Q = R$.

Claim: This mapping is onto. Since $P \mapsto P'$, we may assume that T is a point on m other than P'. By the Euclidean Parallel Property, there is a unique line through T parallel to $\overleftrightarrow{PP'}$. By the argument given above (switching the roles of l and m), this line intersects l in a unique point S. By the definition of our mapping, $S \mapsto T$ (that is, $T = S'$).

3. Given a projective plane $\mathcal{B}$, let l_0 be any line. Let $\mathcal{A}$ consist of all points not on l_0 and all lines not equal to l_0, with the same incidence relation as $\mathcal{B}$ for all the remaining points and lines.

   Claim: $\mathcal{A}$ is an affine plane. Proof of claim: To prove IAI, consider any two points of $\mathcal{A}$. Since $\mathcal{B}$ satisfies IAI, there is a unique line of $\mathcal{B}$ through these points. Since the points to not lie on l_0, this line is not equal to l_0 and is therefore a line of $\mathcal{A}$. To prove IAII, let l be any line of $\mathcal{A}$. Line l is incident with at least three points in $\mathcal{B}$, since $\mathcal{B}$ is a projective plane. Exactly one of the points lying on l is on l_0, by the Elliptic Parallel Property and Proposition 2.1 (which holds for any projective plane, since projective planes are models of incidence geometry; actually, we only need show that at most one point on l is also on l_0, which is true by Proposition 2.1 alone). Thus, at least two points incident with l are points of $\mathcal{A}$. To prove IAIII, note that there must be three non-concurrent lines in $\mathcal{B}$ by Proposition 2.2. At most one of these lines could be l_0, so there are two intersecting lines in $\mathcal{A}$. Since each line has at least two points on it, we obtain three non-collinear points. Finally, to prove the Euclidean Parallel Property, let l be a line of $\mathcal{A}$ and P a point of $\mathcal{A}$ not on l. Let P_0 be the point of intersection of l and l_0. (They intersect by the Elliptic Parallel Property, which holds in $\mathcal{B}$.) Then in $\mathcal{A}$, $\overleftrightarrow{PP_0}$ is parallel to l (since P_0 is not a point of $\mathcal{A}$, and it is the only point where $\overleftrightarrow{PP_0}$ and l intersect in $\mathcal{B}$, by Proposition 2.1). Any other line through P intersects l at a point other than P_0, by the Elliptic Parallel Property and Proposition 2.1. Since this is a point of $\mathcal{A}$, the line is not parallel to l. Thus there exists a unique line through P parallel to l.

   Let $\overline{\mathcal{A}}$ be the projective completion of $\mathcal{A}$. Recall that, in addition to the points and lines of $\mathcal{A}$ with the given incidence relation, $\overline{\mathcal{A}}$ has a point $[l]$ for each equivalence class of lines of $\mathcal{A}$, under the equivalence relation $l \sim m \Leftrightarrow l = m$ or $l \parallel m$, and a line l_∞ incident with all these new points. Each line l intersects l_∞ at $[l]$.

   Claim: $\overline{\mathcal{A}}$ is isomorphic to B. Proof of claim: For each point P of $\mathcal{A}$, let $P' = P$. For each point $P = [l]$, let P' be the point at which l intersects l_0. We must show that this map is well-defined; that is, if $[l] = [m]$, then the points of intersection of l and m with l_0 are the same. This is true because l and m must intersect in $\mathcal{B}$; since they are parallel in $\mathcal{A}$, their unique point of intersection must lie on l_0.

   For each line l of $\mathcal{A}$, let $l' = l$. Let $l'_\infty = l_0$.

   These maps clearly give one-to-one correspondences, and incidence is preserved because $[l]$, the point at which l intersects l_∞, corresponds, by our choice, to the point at which l intersects l_0.

4. Given an affine plane $\mathcal{A}$ with finitely many points, let $\overline{\mathcal{A}}$ be its projective completion. (Note that $\mathcal{A}$ is obtained from $\overline{\mathcal{A}}$ by removing the line l_∞ and all the points on it.)

   Claim: $\overline{\mathcal{A}}$ has only finitely many points. (This fact is needed to use the result of Major Exercise 1 and is not completely obvious.) Proof: We need to show that only finitely many points of the form $[l]$ have been added to $\mathcal{A}$. Since every line is incident with at least two points of $\mathcal{A}$, and there is only one line through any pair of points in $\mathcal{A}$ (IAI), there can be at most as many lines as there are distinct pairs of points. Thus $\mathcal{A}$ has finitely many lines, and hence there are finitely many equivalence classes of lines. (If k is the total number of points, then $\binom{k}{2}$ is the total number of pairs of points.)

Let l and m be any two distinct lines of $\mathcal{A}$. As lines of $\overline{\mathcal{A}}$ they have the same number of points. Each line intersects l_∞ at exactly one point, so has exactly one fewer point lying on it in $\mathcal{A}$. Thus as lines of $\mathcal{A}$ they are also incident with the same number of points.

5. Given a point $P \neq N$ on the sphere, line $\overleftrightarrow{NP}$ intersects the plane Π tangent to the sphere at S, the "South Pole" (that is, the point antipodal to N), at a unique point P'. Conversely, if P' is a point on this plane, the line $\overleftrightarrow{NP'}$ intersects the sphere at a unique point. The mappings $P \to P'$ and $P' \to P$ are clearly inverses of each other, so $P \leftrightarrow P'$ is a bijective correspondence. (This is the slick way to show a mapping is 1-1 and onto!)

   This bijective mapping from the punctured sphere to the plane is called *stereographic projection.* It exists in every dimension, and has many special properties. If you have trouble picturing it for the two-dimensional sphere, try picturing the corresponding map from a circle with one point removed to the line tangent to the antipodal point. For some nice pictures, see the article "A tale of two circles," linked on my Web site.

   This correspondence between points induces a correspondence between the "lines" in the interpretation given and the (actual, straight) lines in the plane Π as follows. A circle on the sphere that passes through N is cut by a plane through N, and the set of points in Π that correspond to those on the circle (other than N, which has been removed) is the intersection with Π of the plane that cuts the circle. The intersection of two planes is a line, so circles on the sphere punctured at N correspond to lines in the plane Π. This correspondence is 1-1 and onto, since for any line in the plane Π, the unique plane containing N and that line cuts a circle through N on the sphere, giving an inverse mapping.

   If a point P is on a circle l through N, then the line $\overleftrightarrow{NP}$ lies in the plane that cuts the circle l (since N and P both lie in this plane); therefore, the point P' corresponding to P lies on the line l' corresponding to l. Similarly, if a point P' lies on line l', the unique plane containing l' and N contains line $\overleftrightarrow{NP'}$, so the point P corresponding to P' lies on the line l corresponding to l'. So the incidence relation is preserved, giving us an isomorphism between the ordinary Euclidean plane and the punctured sphere interpretation. Since the ordinary Euclidean plane is a model of incidence geometry with the Euclidean parallel property, the punctured sphere interpretation must be as well. (We don't even need to check the axioms and Euclidean parallel property directly for the punctured sphere interpretation, although doing so is no harder than what we just did. You just need to know that three points determine a plane, etc.)

6. All you need are two models of incidence geometry, one in which the statement is true, and one in which it is false. You all know two four-point models, one of which is affine, so that there is a 1-1 correspondence between the points on any two lines, and one in which there are lines passing through different numbers of points, so no such correspondence can exist. (This exercise really isn't all that "major"!)

7. (a) Let P be a point. By Proposition 2.4, there is a line l not passing through P. By hypothesis, l is incident with $n+1$ points. Each line through P intersects l at exactly one of these points, by the Elliptic Parallel Property and Proposition 2.1. Furthermore, there is a unique line through P and each point on l (I1). Thus, the points on l and the lines through P are in one-to-one correspondence, and there are $n+1$ lines through P.

   (b) Let P be a point. By I1, every point lies on a line through P. There are $n+1$ distinct lines through P, and each of them is incident with n distinct points different from P. (Except for P, points on different lines are also distinct, since the lines intersect only at P). Thus, counting P there are $(n+1)(n)+1 = n^2+n+1$ points in total.

   (c) To simplify the calculations, let p be the number of points. Every point has $n+1$ lines passing through it, so there are $(n+1)p$ pairs (P, l), where P is a point and l is a line passing through p. Since every line is incident with $n+1$ points, it appears $n+1$ times among these pairs. Thus there are $\frac{p(n+1)}{n+1} = p$ lines, the same number as there are points.

   Here is another argument that some of you may have found: Let l be any line. Every other line intersects l at a unique point, by the Elliptic Parallel Property and Proposition 2.1. Each point on l has $n+1$ distinct lines passing through it, by part (a), and there are $n+1$ distinct points on l. Of course, line l is counted going through each point. Correcting for this we get a total of $(n+1)(n+1)-n = n^2+n+1$ lines. (Of course, you can equivalently just count the lines other than l first and then add l in: $n(n+1)+1 = n^2+n+1$.)

8. You can do this in a similar manner to Exercise 7, or you can use Problem 7 by embedding the affine plane in its projective completion. The latter method is the quickest.

   Given an affine plane $\mathcal{A}$ its projective completion $\mathcal{M}$ has an additional line l_∞, along with all the points on that line. Let n be the number of points in $\mathcal{A}$ on each line of $\mathcal{A}$. In $\mathcal{M}$ each line of $\mathcal{A}$ has an additional point on l_∞; thus, each line of $\mathcal{M}$ has $n+1$ points.

   By Exercise 7, there are therefore $n+1$ lines passing through each point of $\mathcal{M}$, in particular through each point of $\mathcal{A}$. The total number of points in $\mathcal{M}$ is n^2+n+1; subtracting the $n+1$ points on l_∞, we find that there are n^2 points in $\mathcal{A}$. Finally, removing l_∞ from the n^2+n+1 lines of $\mathcal{M}$, we find that there are $n^2+n = n(n+1)$ lines in $\mathcal{A}$.

   Alternatively, here is a proof of part (a) in the manner of 7(a): Let P be a point. By Proposition 2.4, there is a line l not incident with P. Every line through P intersects l *except one.* (By the Euclidean Parallel Property, there is a unique line through P that is parallel to l.) Thus, using the same reasoning as 7(a), the number of lines through P is equal to one more than the number of points on l, or $n+1$. Parts (b) and (c) can be done similarly, but *note that the second method of doing part 7(c) discussed above does not work in the affine context*, and a correct argument that does not use the projective completion is tricky (but possible).

9. A valuable lesson from this problem and the following problem is that **life is better with linear algebra!** I will give, in fact, more than one method of proof using linear algebra for Axiom I-1!

   Proposition 1 (Axiom I-1 in our interpretation)**.** *Given two distinct points (x_1, y_1) and (x_2, y_2), there is a unique line $[u, v, w]$ passing through both of them.*

   Proof. We seek a unique equivalence class (up to non-zero scaling) of ordered triples $[u, v, w]$ such that both points satisfy the equation $ux + vy + w = 0$. In matrix form, this condition can be written as

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

   The only way the two columns of the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ 1 & 1 \end{bmatrix}$ could be multiples of each other is if they were the same, since their last coordinates agree. They cannot be the same, since we have assumed that $(x_1, y_1) \neq (x_2, y_2)$. Thus, this matrix has rank two. Since it represents a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$, its null space must have dimension exactly 1. The non-zero vectors of this null space give an equivalence class $[u, v, w]$ of vectors that are non-zero multiples of each other and thus represent the same line. □

   Here is a second proof using linear algebra:

   Another way to write the equation $ux + vy + w = 0$ is $ux + vy = -w$. In matrix form this is:

$$\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} -w & -w \end{bmatrix}$$

   Case 1. The vectors (x_1, y_1) and (x_2, y_2) are not multiples of each other. In this case, the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$ is invertible, so the only way there is a non-zero vector (u, v) mapping to $(-w, -w)$ is if $w \neq 0$. Up to scaling we may then assume $w = 1$, which uniquely determines u and v. (The vector (u, v) is simply the inverse image of $(-w, -w)$.)

Case 2. The vectors (x_1, y_1) and (x_2, y_2) are multiples of each other; say, $(x_2, y_2) = \lambda(x_1, y_1)$. Then the matrix equation can only be solved if $w = \lambda w$. We know $\lambda \neq 1$, since points (x_1, y_1) and (x_2, y_2) are distinct. Thus, a solution is only possible if $w = 0$. In this case the fact that null space has dimension 1 gives a unique solution for the line through (x_1, y_1) and (x_2, y_2). (It is a line through the origin.)

Proposition 2 (Axiom I-2 in our interpretation)**.** *Given a line $[u, v, w]$, there are (at least) two distinct points satisfying $ux + vy + w = 0$.*

Proof. Since $(u, v) \neq (0, 0)$ by assumption, the equation $\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -w$ has a solution, to which any element of the null space, which has dimension 1, may be added to obtain another solution. This gives infinitely many points satisfying the equation. (A non-zero linear map from $\mathbb{R}^2$ to $\mathbb{R}^1$ must have all of $\mathbb{R}^1$ as its image.) □

Proposition 3 (Axiom I-3 in our interpretation)**.** *There exist three points that do not all satisfy the same linear equation $ux + vy + w = 0$.*

Proof. Every solution to the equation $\begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -w$ is obtained from any given solution by adding an element of the null space. (It is easy to see that any two solutions to a linear equation differ by an element of the null space, since the image of their difference is the difference of their images. Since their images are the same, this difference is zero.) Since the space of points is two-dimensional and the null space is one-dimensional, there must be points that are not obtained in this way and hence are *not* solutions to the equation. □

Proposition 4 (The Euclidean Parallel Property in our interpretation)**.** *Given a line $[u, v, w]$ and a point (x, y) such that $ux + vy + w \neq 0$, there is a line $[u', v', w']$ such that $u'x + v'y + w = 0$ and such that $[u, v, w]$ and $[u', v', w']$ have no common solution.*

Proof. We first claim that two lines $[u, v, w]$ and $[u', v', w']$ have no common solution if and only if (u, v) and (u', v') are multiples of each other, but (u, v, w) and (u', v', w') are not. For if (u, v) and (u', v') are not multiples of each other, the matrix $\begin{bmatrix} u & v \\ u' & v' \end{bmatrix}$ is invertible; hence the equation $\begin{bmatrix} u & v \\ u' & v' \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -w \\ -w' \end{bmatrix}$ must have a solution. The vectors (u, v, w) and (u', v', w') cannot be multiples of each other because, if they were, they would represent the same line.

The unique line $[u', v', w']$ is now obtained by setting $u' = u$, $v' = v$, and $w' = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. □

10. I will first give a short and illuminating proof of each proposition using basic properties of linear maps; then, for completeness, I will give a solution that uses only elementary algebra for all but one of them. (You should be able to fill in the elementary proof I left out if you want to.)

Proposition 5 (Axiom I-1 in our interpretation)**.** *Given two distinct points $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$, there is a unique line $[u, v, w]$ passing through both of them.*

Proof. (*Using linear algebra.*) We seek a unique equivalence class of ordered triples $[u, v, w]$ such that both points satisfy the equation $ux + vy + wz = 0$. In matrix form, this condition can be written as

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

(Here I multiplied the vector we seek, (u, v, w) on the left and the matrix $\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$ on the right, to emphasize duality.)

The map $(u, v, w) \to \begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \\ z_1 & z_2 \end{bmatrix}$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$. Neither column of the matrix $\begin{bmatrix} u & v & w \end{bmatrix}$ is the zero vector, by our definition of point, and these columns are not multiples of each other, since the points $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$ are distinct; therefore, this matrix has rank exactly two. It follows that the dimension of the null space of this map is exactly one. (The sum of the dimensions of the null space and the image space is the dimension of the domain space.) Thus, there is a solution $[u, v, w]$, and all solutions are multiples of each other, therefore in the same equivalence class. □

Proof. (*Not using linear algebra.*)

Case 1. The points $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$ are both nonzero in some coordinate; WLOG, assume this is the first coordinate. We may thus assume that $x_1 = x_2 = 1$.

Subtracting one equation from the other, we get:

$$\begin{array}{rr} & u + vy_1 + wz_1 = 0 \\ - & u + vy_2 + wz_2 = 0 \\ \hline & v(y_1 - y_2) + w(z_1 - z_2) = 0 \end{array}$$

Since the points are distinct, it must be the case that $y_1 \neq y_2$ or $z_1 \neq z_2$; WLOG suppose $y_1 \neq y_2$. Then we can solve for v in terms of w, and then, substituting, for u in terms of w. (I'll leave the details to you. If $[u, v, w]$ is a line, $w \neq 0$, because $w = 0 \Rightarrow v = 0$, and $w = 0 \wedge v = 0 \Rightarrow u = 0$. So we can simplify the calculations by assuming $w = 1$ (scaling u and v accordingly). The line is unique, since any other solution would be a multiple of this one. You then just need to check that each of our two original points satisfies the equation.)

Case 2. In each coordinate, (at least) one of the points is zero. Each point must have at least one nonzero coordinate, which we can scale to 1. Thus, WLOG we may assume $x_1 = 1, x_2 = 0$, $y_1 = 1, y_2 = 0$. One of z_1, z_2 is also zero; WLOG we may assume $z_1 = 0$, although this is not really necessary. *Warning: It is possible that $z_1 = z_2 = 0$, in which case you cannot divide by either of them or by $z_2 - z_1$ (or even by $z_1 + z_2$, although nobody tried that one). A common error was to divide by a quantity that could possibly be zero.*

Subtracting again, we obtain

$$\begin{array}{rr} & u + wz_1 = 0 \\ - & v + wz_2 = 0 \\ \hline & u - v + w(z_1 - z_2) = 0 \end{array}$$

Looking at each equation separately, we can see that we may assume $w \neq 0$. (If it were, so would u and v, so $[u, v, w]$ would not be a line by our definition.) So we may assume $w = 1$ to simplify the calculations. I leave it to you to solve for u and v, giving a unique solution up to equivalence, and check that the solution works.

□

See how much shorter and more insightful the proof was using linear algebra!

Proposition 6 (Axiom I-2+ in our interpretation)**.** *Given a line $[u, v, w]$, there are (at least) three distinct points satisfying $ux + vy + wz = 0$.*

Proof. (*Using linear algebra.*) Consider the matrix equation

$$\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

By our definition of line, $[u, v, w] \neq [0, 0, 0]$; thus the matrix $\begin{bmatrix} u & v & w \end{bmatrix}$ has rank 1. The map taking $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ to $\begin{bmatrix} u & v & w \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a linear map from $\mathbb{R}^3$ to $\mathbb{R}$. Thus, its null space has rank 2. This means there are two independent vectors $[x_1, y_2, z_1]$ and $[x_2, y_2, z_2]$ satisfying the equation. Let $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$, $z_3 = z_1 + z_2$. Then none of the vectors $[x_1, y_2, z_1]$, $[x_2, y_2, z_2]$ and $[x_3, y_3, z_3]$ is a multiple of any other (representations as linear combinations of independent vectors are unique), so they represent three distinct points on the given line.

□

Proof. (*Not using linear algebra.*)
WLOG, assume $u = 1$. (Why is this permissible?) It is easy to see that, for example, $[-w, 0, 1]$, $[-v, 1, 0]$, and $[-v - w, 1, 1]$ are distinct (why?) points satisfying the equation. □

In this case, linear algebra is illuminating, giving a simple reason that the solutions must exist and unifying all the proofs with the key concepts of matrix rank and dimension, but does not shorten the proof, since the direct calculation is so easy.

Proposition 7 (Axiom I-3 in our interpretation)**.** *There exist three points that do not all satisfy the same homogeneous linear equation* $ux + vy + wz = 0$.

Proof. (*Using linear algebra.* Let $[u, v, w]$ be any line. We know there are two distinct points on it by Proposition 2. We need a third point that is not on this line.
Referring to the linear map in the previous proof, since $[u, v, w]$ is not the zero vector (and thus has rank 1), not all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ are taken to 0. Therefore, there is a point *not* satisfying $ux + vy + wz = 0$. (Here we really needed that $[u, v, w] \neq [0, 0, 0]$. In the proof of the previous proposition, we only really needed that the dimension of the image was at least two less than that of the domain. Here we need that the dimension of the image is not 0.) □

Proof. (*Not using linear algebra.*) Consider, for example, the points $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$. (This is the most "symmetric" example.) These points are clearly distinct (why?). The first two are on the line $z = 0$ (that is, the line given by the triple $[0, 0, 1]$ - isn't that cute?); the third is clearly not.

□

Proposition 8 (The Elliptic Parallel Property in our interpretation)**.** *Any two equations* $u_1x + v_1y + w_1z = 0$ *and* $u_2x + v_2y + w_2z = 0$ *have a common solution.*

Proof. Consider the matrix equation

$$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

You should be able to complete the proof! What is the rank of the matrix $\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{bmatrix}$? Why? What does this tell you about the null space of the linear map given by this matrix?

□

Remark. You also get Proposition 2.1 from this argument, basically for free, although we know it can also be proven from the incidence axioms.

We are also asked to verify that the real projective plane (described in this problem) is the projective completion of the real affine plane (described in the previous problem). We identify each affine point (x, y) with the projective point $[x, y, 1]$ and each affine line $[u, v, w]$ (representing the equation $ux + vy + w = 0$) with the projective line $[u, v, w]$ (which represents a different equation, the homogeneous equation $ux + vy + wz = 0$ in three variables). With these identifications, the incidence relations of the affine plane are preserved, since $ux + vy + w = ux + vy + w \cdot 1$. (When we say we are "identifying" two things, what we really mean precisely is we are setting up an isomorphism in which they correspond.)

Observe that for an affine line $[u, v, w]$, we may never have both $u = 0$ and $w = 0$. Thus, this case, represented by the equivalence class $[0, 0, 1]$ is a new line, representing the equation $z = 0$. As noted in the text, this is the line at infinity.

In addition, since each affine point (x, y) is identified with the projective point $[x, y, 1]$, the points of the form $[x, y, 0]$ are new. They are the points at infinity. Since $z = 0$ for these points, they lie on the line at infinity.

The affine line $[u, v, w]$ intersects the line at infinity, $z = 0$, at the point $[v, -u, 0]$. As expected, this point does depend only on the ratio between $-u$ and v, that is on the slope of the line. (We allow this ratio to be undefined - "$\frac{1}{0}$", for a vertical line.) For fixed values of u and v (up to scaling) and varying values of w, all of the parallel lines $[u, v, w]$ pass through this same point in the projective completion. (When you plug in 0 for z, the value of w does not matter.) There is one point at infinity for each slope, and different slopes give different projective points. (Why?) This completes the verification that the real projective plane is the projective completion of the real affine plane.

Finally, let us see why the real projective plane is isomorphic to the models of Exercises 9(c) and 9(e) (see page 64). I leave it to you to show that these two models are isomorphic; I show below that the real projective plane is isomorphic to one of them.

Consider the model of 10(c), in which "points" are lines through the origin, "lines" are planes through the origin, and a "point" is incident with a "line" if the line defining that "point" lies in the plane defining that "line." Each point $[x, y, z]$ of the real projective plane is the set of all points in $\mathbb{R}^3$ other than $(0, 0, 0)$ lying on the line through the origin that is the span of the vector (x, y, z) (that is, the line consisting of all multiples of this vector), and different projective points give different lines (since their representative points are not multiples of each other). So taking $[x, y, z]$ to the line through $(0, 0, 0)$ and (x, y, z) gives a 1-1 and onto correspondence between the points of the real projective plane and the points of the model of 10(c).

Similarly, taking each projective line $[u, v, w]$ to the plane through the origin defined as the solution to the equation $ux + vy + wx = 0$ gives a 1-1 and onto correspondence. (I leave it to you to check the details.) A projective point $[x, y, z]$ lies on the line $[u, v, w]$ if and only if the line through $(0, 0, 0)$ and (x, y, z) lies in the plane defined by $ux + vy + wz = 0$. (Again I leave it to you to verify the details.) So the incidence relation is preserved, and therefore this correspondence is an isomorphism.