MAT 3271: Selected Solutions to the Assignment 5

Chapter 2 (Exercises)

8. First let us verify that the 4-point model of Example 3 satisfies the incidence axioms. We can do this in general, without listing every case, if we note that the set of lines is exactly the set of all subsets of the set of points having exactly two elements. For Incidence Axiom I, let P and Q be any two distinct points (chosen from the set $\{A, B, C, D\}$). The set $\{P, Q\}$ is a line incident with both these points, and there is no other line incident with both of them because any other set with exactly two elements would have to exclude either P or Q. Incidence Axiom II is satisfied because every line is a set of two elements. Incidence Axiom III is satisfied because there is no line with three elements, so any three distinct points are non-collinear. To prove the Euclidean Parallel Property, let l be any line and let P be a point not on l. By definition (of *line* and *incidence* in this interpretation), $l = \{R, S\}$ for two distinct points R and S, chosen from the set $\{A, B, C, D\}$, each of which is also distinct from P. Thus there remains exactly one point, Q, distinct from P, R, and S, and $\{P, Q\}$ is the unique line through P and parallel to l.

   For the 5-point model of Example 4, the proofs that the incidence axioms hold are the same. To prove the Hyperbolic Parallel Property, let l be any line and let P be a point not on l. As in the previous model, $l = \{S, T\}$ for two distinct points S and T, chosen from the set $\{A, B, C, D, E\}$, each of which is also distinct from P. In this model, in which the set of points has five elements, there remain two distinct points, Q and R, each of which is distinct from P, S, and T. Thus $\{P, Q\}$ and $\{P, R\}$ are distinct lines through P that are parallel to l. [Note that simply showing that a given line has more than one line parallel to it is not sufficient to prove the Hyperbolic Parallel Property; the lines must be shown to pass through a common point.]

10. (a) This exercise is more abstract than the proof in the text that the two given 3-point models are isomorphic. We must make no assumptions about what the models are. They need not have any particular set-theoretic interpretation, in which, for example, lines are sets with two elements, or points are sets of two elements. Incidence need not be defined by set membership. There are many other conceivable models of incidence geometry with three points. (Example: Points are the vertices of an equilateral triangle; lines are reflections that take the triangle to itself; a point is incident with a line if it is moved by the reflection.) We must show that *any* two such models are isomorphic.

    What we are given is simply that both interpretations are models of incidence geometry, which means that the three incidence axioms hold for them. We must use these axioms to establish how many lines there are and what points are incident with each line. In addition, we are given and can use (indeed, must use) that each model has three points, however these points might be defined. That's it!

    With these ideas in mind, let the points of Model 1 (whatever they are) be labeled P, Q, and R, and those of Model 2 be labeled P', Q', and R'. By Incidence Axiom I, there must be lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{PR}$ in Model 1, and no line other than one of these passes through any pair of points. By Incidence Axiom II, every line is incident with at least two points, and by Incidence Axiom III, no line passes through P, Q, and R; therefore, $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{PR}$ are distinct and are the only lines in Model 1. Similarly, $\overleftrightarrow{P'Q'}$, $\overleftrightarrow{Q'R'}$, and $\overleftrightarrow{P'R'}$ are the lines of Model 2. The isomorphism between these models should now be obvious.

    (b) If you look among the models already described in the text and in class, you will find a pair of 4-point models that are not isomorphic. (They cannot be isomorphic because they have different numbers of lines. They also differ in the number of points incident with a given line.)

    (c) Before checking the isomorphism, let us make sure we understand the two models. In the sphere model, the *points* are not individual points on the sphere but pairs of antipodal points. You should now recognize these pairs as equivalence classes, where the equivalence relation on the set of points is that $P \sim Q \Leftrightarrow [P = Q$ or P is antipodal to $Q]$. It should be obvious that this relation is reflexive, symmetric, and transitive. (Note that if $P \sim Q$ and $Q \sim R$, then two of the points must be equal.)

Note also that every circle, by definition, lies in a plane, and thus every circle on the sphere is the intersection of the sphere with a plane. The *great circles* are, by definition, those circles whose center is O; that is, they are the circles at which planes through the origin intersect the sphere. (These are clearly the circles on the sphere with the largest radius, which is equal to the radius of the sphere. Why?)

The reason Incidence Axiom 1 is satisfied is that any two points that are not antipodal are also not collinear with the center of the sphere, O. Thus, if P and Q are not antipodal, P, Q, and O determine a unique plane, which cuts of a great circle through P and Q. The points antipodal to P and Q are automatically on this great circle, since if a plane contains two points on a line it contains the whole line. (So, for example, since P and O are on the line, so is the collinear point P' antipodal to P.) Incidence Axioms 2 and 3 should be obvious. (There are infinitely many points on each great circle, and the sphere does not lie in a single plain, so not all points on it are collinear.)

For the projective spatial model in which points are lines through the origin and lines are planes through the origin, Incidence Axiom 1 is satisfied because two distinct intersecting lines determine a unique plane containing both of them. (Why?) Similarly to the case of the sphere model, Incidence Axioms 2 and 3 are obvious.

Note that the elliptic parallel property holds in both models: Any two planes through the origin intersect in a line through the origin, and any two great circles on the sphere, being cut by planes through the center, intersect at a pair of antipodal points, where the line at which the planes intersect cuts through the sphere. These are both models of the geometry called the *real projective plane*.

The obvious isomorphism, as indicated by the hint, is to place the center of the sphere at the origin and map each line through the origin to the pair of antipodal points at which it intersects the sphere and each plane through the origin to the great circle in which it intersects the sphere. (This map is clearly one-to-one and onto, since any pair of antipodal points, being the endpoints of a diameter, determines a line through the origin, and any great circle is the intersection with the sphere of a unique plane through the origin.) Most of you got this far, but there is a little more to do: we must check that the incidence relation is preserved. Suppose a line l through the origin (which is a *point* of the first model) lies in plane π through the origin (which is a *line* of the first model that is incident with that *point*). Then the intersection of l with the sphere is a pair of antipodal points on the great circle in which the whole plane π intersects the sphere. This means that the image of l under this isomorphism is incident with the image of π. To show that the incidence relation is completely preserved, we must also show the converse. (If a point and line in one model are *not* incident, their images under an isomorphism cannot be incident either.) So suppose a pair of antipodal points lies on a great circle; then, as previously discussed, the line through these antipodal points lies in the plane cutting off the great circle. Thus, the corresponding point and line in the projective spacial model are incident, and we have completed the proof.

11. Note that, as we are considering them here, the elliptic, Euclidean, and hyperbolic parallel properties are *universal*: their defining properties must hold for *any* line l and *any* point P not on l. Therefore, if there is even one exception, they fail.

    Consider the model in which the set of points is the five-element set $\{A, B, C, D, E\}$, the set of lines is the set $\big\{\{A, B, C\}, \{A, D\}, \{A, E\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\big\}$, and incidence is defined by set membership. (I leave it to you to draw a diagram; it simply involves “collapsing” two lines of the five-point hyperbolic model into one.) This model is neither elliptic nor hyperbolic, because there is a unique line through D parallel to $\{A, B, C\}$, namely $\{D, E\}$. It is not Euclidean, because there are two distinct lines through E that are parallel to $\{B, D\}$: $\{A, E\}$ and $\{C, E\}$.