MAT 3271: Selected Solutions to the Assignment 4

Chapter 2

Exercises (as opposed to *major* exercises)

3. There exist two right angles that are not congruent.

4. There exist a line l and a point P not incident with l such that there is no line through P that is parallel to l or there exist two distinct lines through P that are parallel to l.

5. (a) If a transversal t to lines l and m cuts out congruent alternate interior angles, then l and m are parallel. [Note: "cuts out" and "alternate interior angles" will have to be defined.]

   (b) If lines l and m meet on one side of a transversal t, then the sum of the interior angles on that side of the transversal is less than $180°$.

6. Your proofs may vary from the ones below and still be correct.

Proposition 2.1. If l and m are distinct lines that are not parallel, the l and m have a unique point in common.

Proof. Since l and m are not parallel, by definition they have a point of intersection, call it P. Suppose l and m also intersect at a point Q distinct from P. Then by Incidence Axiom 1 (uniqueness part), $l = m$, contradicting the hypothesis that they are distinct. Thus l and m have a unique point of intersection. $\square$

Proposition 2.2. There exist three distinct lines that are not concurrent.

Proof. By Incidence Axiom 3, there exist three non-collinear points, call them P, Q and R. By Incidence Axiom 1, there exist lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{QR}$, and $\overleftrightarrow{RP}$. These three lines are distinct, by the definition of non-collinear. Since $\overleftrightarrow{QR}$ and $\overleftrightarrow{RP}$ intersect at R, and since their point of intersection is unique by Proposition 2.1, the three lines are concurrent only if line $\overleftrightarrow{PQ}$ passes through R. It does not, again because P, Q, and R are not collinear. $\square$

Proposition 2.3. For every line, there is at least one point not lying on it.

Proof. Let l be a line. Consider the three non-collinear points given by Incidence Axiom 3. By definition, they cannot all lie on l. Thus there is a point not lying on l. $\square$

Proposition 2.4. For every point, there is at least one line not passing through it.

Proof. Let P be a point. By Proposition 2.2, there are three lines that are not concurrent. By definition, they cannot all pass through P. So there is a line not passing through P. $\square$

Proposition 2.5. For every point P there exist at least two lines through P.

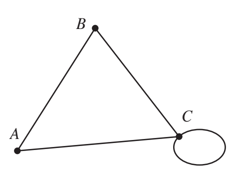
Proof. Let P be a point, and consider once again the three non-collinear points, let's call them Q, R and S, given by Incidence Axiom 3. (Note that it is possible that P is one of these points, and possible that it isn't.)

Case 1: P is one of the points Q, R or S. Without loss of generality, suppose $P = Q$. Then lines $\overleftrightarrow{PR}$ and $\overleftrightarrow{PS}$ (given by Incidence Axiom 1) are distinct, by the definition of non-collinearity.

Case 2: Consider the lines $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, and $\overleftrightarrow{PS}$. At least two of these lines must be distinct, because if they are all the same line, then Q, R, and S would all lie in it, again contradicting the fact that they are non-collinear. $\square$

7. In the following model, Incidence Axioms I and II are satisfied, but not III: Let the set of points be $\{A, B\}$ and the set of lines be $\{\{A, B\}\}$, with incidence given by set membership. In other words, there is one line with two points, A and B, lying on it.

In the following model, Incidence Axioms I and III are satisfied, but not II: Let the set of points be $\{A, B, C\}$ and the set of lines be $\{\{A, B\}, \{B, C\}, \{C, A\}, \{C\}\}$. In other words, there are three non-collinear points A, B, and C, and in addition to the lines required by Incidence Axiom I, there is a line with one point (C) on it.



In the following, model, Incidence Axioms II and III are satisfied, but not I: Let the set of points be $\{A, B, C\}$ and the set of lines be $\{\{A, B\}, \{B, C\}\}$. Thus, the existence requirement of Incidence Axiom I is violated. (There is no line through A and C.) Alternatively, we could violate the uniqueness requirement of Incidence Axiom I by choosing the set of points to be $\{A, B, C, andD\}$ and the set of lines to be $\{\{A, D, B\}, \{A, C, B\}, \{C, D\}\}$. I leave it to you to draw diagrams representing these models.

You may have chosen different models. For example, the empty model, with no points and no lines, vacuously satisfies Incidence Axioms 1 and 2, but not 3. (Anything is true for all elements of the empty set; after all, there does not exist any exception!)