MAT 3271: Selected Solutions to Assignment 3

1 Chapter 1

7. To show that $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$ is easy; the definition of ray says in general that the points of ray $\overrightarrow{PQ}$ lie on line $\overleftrightarrow{PQ}$. If it didn't say that, we'd need an axiom saying that if a point is between two other points, then all three points are collinear. We will indeed state such an axiom.

   The more subtle part of this problem is to show that $\overleftrightarrow{AB} \subseteq \overrightarrow{AB} \cup \overrightarrow{BA}$. The point is, how do you prove that every point on the line is in one ray or the other? To do this from the definition of ray, we need to know that any point on line $\overleftrightarrow{AB}$ not equal to A or B must be in *some* "betweenness relationship" with A and B: that is, if P lies on $\overleftrightarrow{AB}$, then $P * A * B$ or $A * P * B$ or $A * B * P$. Each case then fits the definition of one ray or the other (or both). It is also necessary to know that $P * Q * R \Leftrightarrow R * Q * P$; we will indeed an axiom to that effect.

   To prove that $AB \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$ is, of course, easy since the segment is a subset of each ray by definition. (Here we again need to know that $P * Q * R$ means the same as $R * Q * P$ in order to show that $AB = BA$.) To show the converse inclusion requires that $P * A * B$ and $A * B * P$ be mutually exclusive, so that only $A * P * B$ is possible if $P \neq A$ and $P \neq B$. Thus, we need to strengthen our axiom for three (distinct) collinear points P, Q, and R to say that *exactly* one of $P * Q * R$, $Q * R * P$, or $R * P * Q$ holds.

   Carefully thinking about these questions allows us to anticipate the betweenness axioms, which will be introduced in Chapter 3!

8. (i) $\Rightarrow$ (ii) follows from Euclid's fourth postulate (which we will actually prove as a theorem): all right angles are congruent. (ii) $\Rightarrow$ (i) requires the assertion that the angle sum of any convex quadrilateral is $360°$. (This assertion is a theorem of Euclidean geometry, but is false in non-euclidean geometry.) Of course, we also need to establish that the measure of a right angle is $90°$, but that is just a matter of convention that establishes the size of a degree. (i) $\Rightarrow$ (iii) follows from the theorem that, if the sum of the interior angles on one side of a transversal is $180°$, then the lines are parallel. (This theorem is valid in both Euclidean and hyperbolic geometry.) (iii) $\Rightarrow$ (i) requires the converse of that theorem: if any two parallel lines are cut by a transversal, then the the sum of the interior angles on one side of a transversal is $180°$. (The converse is a theorem of Euclidean geometry but is false in hyperbolic geometry.) Note that we do not have to separately prove that (ii) $\Leftrightarrow$ (iii), since implication is transitive.

12. Think about the last sentence of the proof!

13. We would need to prove that for any two circles, the ratios of circumference to diameter are equal. (We would also have to define length, including for curves, so we could measure the circumference and diameter; as you know, the length of a curve involves integral calculus.)