MAT 3271: Selected solutions to problem set 2

Chapter 1, Exercises:

1. (a) The *midpoint* of segment AB is the point $M \in AB$ such that $AM \cong MB$. (Recall that "$\cong$" stands for "is congruent to." Use of the word "the" is pending proof, which we will supply, that the point satisfying this definition is unique. It is implicit that M is distinct from A and B; otherwise, segments AM and MB would not be defined. But if you prefer, you can state that M is between A and B to make clear that it is not equal to either of them.)

   (b) The *perpendicular bisector* of segment AB is the line incident with the midpoint of AB that is perpendicular to line $\overleftrightarrow{AB}$. (Again, use of the word "the" is pending proof that the perpendicular through a given point is unique.)

   (c) Given that D is between A and C, ray $\overrightarrow{BD}$ bisects angle $\angle ABC$ if $\angle ABD \cong \angle CBD$. (Note that this definition does not depend on any specific way to check, such as that $BA \cong BC$ and $AD \cong CD$. The reason we should not do that is that angles depend only on rays, not on specific points on the rays that are used to name them. Moving point A on ray $\overrightarrow{BA}$ so that $BA \not\cong BC$, for example, would not change whether or not $\overrightarrow{BD}$ bisects angle $\angle ABC$. Make the conditions in your definitions only as restrictive as necessary, but no stricter.)

   (d) Points A, B, and C are *collinear* if there is a common line incident with all of them. ("Common" means that it is one line, not a possibly different line incident with each point. There is really no reason to name the points, since we don't need to use the names. So we could just say that three points are collinear if .... And we can easily extend the definition to any number of points. It is not necessary to say the points are distinct, since fewer points would still lie on a common line, but the term *collinear* is only meaningful for three or more points, since any two points are incident with a common line.)

   (e) Lines l, m, and n are *concurrent* if there is a common point incident with all of them. (Again, we don't really need to give the lines names, and we do not need to restrict attention to a specific number of lines. If there are only two lines, we generally say that they *intersect* rather than that they are concurrent. Any two lines that are not parallel intersect, by definition of not being parallel; however, there is no reason that three lines have to intersect at a common point, even if no two are parallel. As with the definition of *collinear*, it is not necessary to say the lines are distinct, but as just noted the term is customarily used only for three or more distinct lines.)

2. (d) A *median* of a triangle is a segment whose endpoints are a vertex and the midpoint of the side opposite that vertex.

   [Comment: The side opposite a vertex was defined in Exercise 2(c). We also say that the vertex is opposite the side. The angle opposite a side is understood to mean the angle whose vertex is opposite that side.]

   (e) To define altitude, it is helpful to first define the *foot* of a perpendicular:

   Given a line l and a point P not incident with l, the *perpendicular* from P to l is the line through P that is perpendicular to l. [Comment: Our use of the word "the" is justified by the fact that we will prove there exists a unique line through P that is perpendicular to l.] The *foot* of the perpendicular from P to l is the point at which it intersects l.

An *altitude* of a triangle is a segment whose endpoints are a vertex and the foot of the perpendicular from that vertex to the line through the other two vertices.

[Comment: Note the careful distinction made between lines and segments. The foot of the perpendicular may not be in the side of the triangle opposite the vertex.]

(f) A triangle is *isosceles* if two of its sides are congruent. Given two specified sides that are congruent, the remaining side is called the *base* of the triangle. The *base angles* of the triangle are the angles whose vertices are the endpoints of the base.

[Comments: Equivalently, the base angles are the angles opposite to the specified congruent sides. Note that the base of an isosceles triangle may be congruent to the other two sides: the set of equilateral triangles (defined next) is a subset of the set of isosceles triangles. Therefore, I think it is really best to speak of *the isosceles triangle* $\triangle ABC$ *with base* BC *and apex* A, meaning that sides AB and AC are specified to be congruent.

Although we can prove that the base angles of an isosceles triangle are congruent in both the Euclidean and noneuclidean classical geometries, this fact is not part of the definition. We would still call the triangle isosceles if it has a pair of congruent sides, even if the base angles were not congruent, and we can easily give an example of a nonclassical geometry - one whose curvature is not uniform - in which they are not.]

(g) A triangle is *equilateral* if all of its sides are congruent.

[Comment: Although we can prove that the angles of an equilateral triangle are all congruent (that is, the triangle is equiangular) in both the Euclidean and noneuclidean classical geometries, this fact is not part of the definition. On a surface whose curvature is not uniform, we can easily demonstrate a triangle that is equilateral but not equiangular.]

4. The shortest definition I know of vertical angles is: Two angles are *vertical* angles if each is supplementary to a common angle.

(Here the term supplementary is used in the specific sense of the definition on page 17 of the text, not in the more general sense of having measures adding to 180°. If desired, one can avoid using the word "supplementary" as follows:

Two angles are *vertical* angles if each side of one is opposite to a side of the other. [Comment: The sides of an angle are rays, and what it means for rays to be opposite was defined in the text.])

Alternatively, you can label five points and refer to the labels, but then you have to be very careful to completely describe the points in words without reference to a picture. A picture may be added for illustration, but it is not part of the definition. Thus:

Let distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at point E, where E is between A and B ($A * E * B$, for short) and also between C and D. Then angles $\angle AEC$ and $\angle BED$ are a pair of vertical angles, as are angles $\angle AED$ and $\angle BEC$.

[Comment: Note that you can construct the picture from the description. That you can do so is a sign of a good definition.]