MAT 3271: Geometry **Name:** ________________________________

Final Exam: take-home portion. Due at the beginning of the in-class final exam.

You are expected to work on this exam alone and to refrain from talking about the exam to anyone except the professor until the time and date when it is due. You may use your own notes and any published materials that you like. Be sure to cite any published sources that you use.

Your signature below attests to a pledge that you have done the exam according to the above instructions. You must attach a signed cover page to your solutions.

Signature:________________________________

Recall the proof of the Saccheri-Legendre Theorem of neutral geometry, which states that the sum of the degree measures of the angles of any triangle is less than or equal to 180°: Suppose by way of contradiction that there exists a triangle $\triangle ABC$ whose angle sum is greater than 180°. It is claimed that we can replace $\triangle ABC$ with a new triangle whose angle sum is the same, but such that one of its angles has at most half the degree measure of $\angle A$. By repeating this procedure we eventually obtain a triangle with one angle so small that the degree measures of the other two angles must sum to at least 180°, which violates the Corollary to the Exterior Angle Theorem. (Note that here we are using the consequence of Archimedes' Axiom that you proved in problem 4 of Exam 3.) Your task for the first problem of this exam is to prove the claim just mentioned.

1. Given $\triangle ABC$, there exists a triangle with the same angle sum as $\triangle ABC$ such that one of its angles has at most half the measure of $\angle A$. (Hint: Use the construction in the first problem of Exam 3: Given $\triangle ABC$, let D be the midpoint of BC, and let E be the point on the ray opposite $\overrightarrow{DA}$ such that $DE \cong DA$. Use congruent triangles and angle addition to show the angle sums of $\triangle ABC$ and $\triangle AEC$ are equal. Furthermore, you proved that $\angle E \cong \angle EAB$. Thus, $(\angle E)^\circ + (\angle EAC)^\circ = (\angle BAC)^\circ$. If the sum of two numbers equals a third number, they cannot both be greater than half of it! Note that you cannot tell which angle is smaller, if one is, and it is even possible that they they both have measure exactly equal to half the measure of $\angle A$, but it is not necessary to know which angle it is, only that one of the angles of $\triangle EAC$ has at most half the measure of $\angle A$.)

It is possible to state results such as the the triangle inequality, the corollary to the Exterior Angle Theorem, and even the Saccheri-Legendre Theorem without introducing numbers. We do this by instead introducing an addition operation on congruence classes of segments and on congruence classes of angles. (Note that the sum of two congruence classes of angles will not always be defined, because we have not introduced the concept of an angle that is "greater than a straight angle.") The introduction of these operations also makes it easier to state Archimedes' Axiom, which is necessary for the measurement of segments and angles by real numbers. (In addition, note that the *proof* of the Saccheri-Legendre Theorem *does* require Archimedes' Axiom, even though its statement does not.)

Recall that congruence is an equivalence relation, so it partitions the set of all segments into equivalence classes; all of the segments in each equivalence class are congruent to each other, but not congruent to the segments in any other equivalence class. We call these equivalence classes *congruence classes* on account of the particular equivalence relation they come from. We define the sum of two congruence classes as follows:

Definition (for segments)**.** Given two congruence classes of segments, $[AB]$ and $[CD]$, let E be the point on the ray opposite $\overrightarrow{BA}$ such that $BE \cong CD$. (Point E exists and is unique by Congruence Axiom 1.) We define the sum of $[AB]$ and $[CD]$ by $[AB] + [CD] = [AE]$.

Definition (for angles)**.** Given two congruence classes of angles, $[\angle ABC]$ and $[\angle DEF]$, let G be a point on the side of $\overleftrightarrow{AB}$ opposite C such that $\angle GBA \cong \angle DEF$. (Although point G is certainly not unique, ray $\overrightarrow{BG}$ exists and is unique by Congruence Axiom 4.) If G lies on the same side of $\overleftrightarrow{BC}$ as A, we say the sum of $[\angle ABC]$ and $[\angle DEF]$ *is less than a straight angle* and define it by $[\angle ABC] + [\angle DEF] = [\angle GBC]$. If angles $\angle ABC$ and $\angle GBA$ are supplementary, we say the sum of $[\angle ABC]$ and $[\angle DEF]$ *is a straight angle.* (If G lies on the opposite side of $\overleftrightarrow{BC}$ from A, the sum of $[\angle ABC]$ and $[\angle DEF]$ is not defined.)

These definitions involve arbitrary choices: for example, we chose a *particular* segment AB out of its congruence class and a particular segment CD out of its congruence class; moreover, we implicitly chose a direction for AB by marking off point E on the ray opposite $\overrightarrow{BA}$ rather than the ray opposite $\overrightarrow{AB}$. Analogous choices were made in the case of angles. For these definitions to be legitimate – that is, for the result of these summing operations to be *well-defined* – we must show that the congruence class of the result does not depend on these choices. To do so is your task on the next two problems of this exam!

2. Suppose $A'B' \cong AB$ and $C'D' \cong CD$. Let E be the point on the ray opposite $\overrightarrow{BA}$ such that $BE \cong CD$, and let E' be the point on the ray opposite $\overrightarrow{B'A'}$ such that $B'E' \cong C'D'$. Prove that $A'E' \cong AE$; hence, addition is well-defined on congruence classes of segments. (Hint: This is a straightforward application of the segment addition proposition. Remark: The choice of direction of segment AB is covered by this argument, since $AB \cong BA$.)

3. Suppose $\angle A'B'C' \cong \angle ABC$ and $\angle D'E'F' \cong \angle DEF$. Let G be a point on the side of $\overleftrightarrow{AB}$ opposite C such that $\angle GBA \cong \angle DEF$, and let G' be a point on the side of $\overleftrightarrow{A'B'}$ opposite C' such that $\angle G'B'A' \cong \angle D'E'F'$. For angles the situation is a bit more complicated, since there are three possibilities:

   (a) Prove that G' lies on the same side of $\overleftrightarrow{B'C'}$ as A' if and only if G lies on the same side of $\overleftrightarrow{BC}$ as A. Thus the sum of $[\angle A'B'C']$ and $[\angle D'E'F']$ is less than a straight angle if and only if the sum of $[\angle ABC]$ and $[\angle DEF]$ is less than a straight angle. (Hint: Use Propositions 3.8(c), one of the lemmas we proved in class, and the uniqueness part of Congruence Axiom 4.)

   (b) Prove that angles $\angle A'B'C'$ and $\angle G'B'A'$ are supplementary if and only if angles $\angle ABC$ and $\angle GBA$ are supplementary. Thus, the sum of angles $\angle A'B'C'$ and $\angle D'E'F'$ is a straight angle if and only if the sum of angles $\angle ABC$ and $\angle DEF$ is a straight angle. (Hint: This is an application of the proposition that supplements of congruent angles are congruent, but you also have to use the uniqueness part of Congruence Axiom 4, in a similar manner to what we have done on several occasions before.)

   (c) Finally, prove in the case that the sum of each pair of angles is less than a straight angle that $[\angle ABC] + [\angle DEF] = [\angle A'B'C'] + [\angle D'E'F']$. (Hint: This is a straightforward application of the angle addition proposition.)

I invite you to consider how to prove that these addition operations are commutative and associative. The proofs are not difficult, but somewhat tedious, so I have not included them on the exam. The sum of more than two segments or angles is defined recursively in the obvious way, as is $n[AB]$, where n is a natural number. Finally, by the segment and angle ordering propositions (parts (b) and (c) of Propositions 3.13 and 3.21), order is well-defined on equivalence classes.

With these definitions, we can restate some of our theorems and axioms without invoking measurement:

Corollary (corollary to the Exterior Angle Theorem)**.** *The sum of any two angles of a triangle is less than a straight angle.*

Theorem (Saccheri-Legendre Theorem)**.** *The sum of the angles of any triangle is less than or equal to a straight angle.*

Archimedes' Axiom becomes: Given a unit segment UT and a segment AB, there is a natural number n such that $n[UT] \geq [AB]$.

4. Restate the Triangle Inequality (see Greenberg, p. 124) without using length. (Yes, this is as short and easy as it looks!)

The final problem of the exam makes use of the Hilbert's Euclidean Parallel Postulate. Recall its statement: Given a line l and a point P not on l, there is at most one line through P that is parallel to l. (The existence of a line through P that is parallel to l follows from the Alternate Interior Angle Theorem.) If we add the Euclidean Parallel Postulate to the axioms of neutral geometry, we get Euclidean Geometry. (If we add the negation of the Euclidean Parallel Postulate instead, we get hyperbolic geometry, which you will study in MAT 3272.)

5. (a) Prove (in neutral geometry) that the Euclidean Parallel Postulate is equivalent to the converse of the Exterior Angle Theorem. (Hint: To prove the Euclidean Parallel Postulate from the converse to the Alternate Interior Angle Theorem, use the uniqueness part of Congruence Axiom 4. To prove the converse of the Alternate Interior Angle Theorem from the Euclidean Parallel Postulate, use the fact that you can always construct a parallel line with congruent alternate interior angles, and then apply the uniqueness given by the Euclidean Parallel Postulate.)

   (b) Prove that in Euclidean geometry, the angle sum of any triangle is equal to $180°$. (Hint: Given $\triangle ABC$, let l be the unique line through A that is parallel to $\overleftrightarrow{BC}$ and use the converse of the Alternate Interior Angle Theorem.)