Axioms of the Real Number System

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1 Axioms of Addition and Multiplication

There exist binary operations $+$ and $\cdot$ on $\mathbb{R}$, such that:

1. These operations are *associative*:
   (a) $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$.
   (b) $\forall x, y, z \in \mathbb{R}, (x \cdot y) \cdot z = x \cdot (y \cdot z)$.
2. These operations are *commutative*:
   (a) $\forall x, y \in \mathbb{R}, x + y = y + x$.
   (b) $\forall x, y \in \mathbb{R}, x \cdot y = y \cdot x$.
3. Each operation has a distinct *identity element*:
   (a) There exists an element $0 \in \mathbb{R}$ such that, $\forall x \in \mathbb{R}$, $x + 0 = x$.
   (b) There exists an element $1 \in \mathbb{R}$ such that $1 \neq 0$ and, $\forall x \in \mathbb{R}$, $x \cdot 1 = x$.
4. All possible *inverses* exist:
   (a) For each x in $\mathbb{R}$, there exists a y in $\mathbb{R}$ such that $x + y = 0$.
   (b) For each x in $\mathbb{R}$ different from 0, there exists a y in $\mathbb{R}$ such that $x \cdot y = 1$.
5. The operation $\cdot$ *distributes* over $+$:
   $\forall x, y, z \in \mathbb{R}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

2 Axioms of Order with Respect to the Operations

There is a total order relation $<$ on the real number system with the following properties:

6. The operations respect the order relation:
   (a) If $x > y$, then $x + z > y + z$.
   (b) If $x > y$ and $z > 0$, then $x \cdot z > y \cdot z$.
7. The order relation $<$ has the *least upper bound property*.

Remark. We have note used the last axiom, and you do not need it for any exam question except the extra credit question on the second take-home exam.