Two Elementary Area Theorems: Pythagorean Theorem & Area of a Circle

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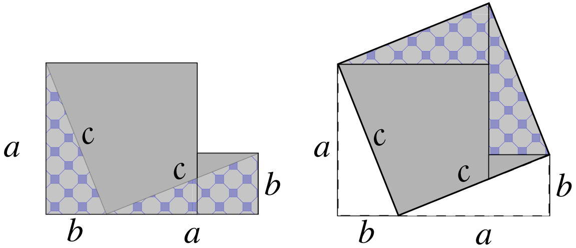
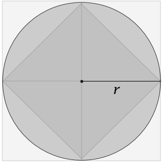
Why the Pythagorean Theorem is true

$$a^2 + b^2 \quad = \quad c^2$$

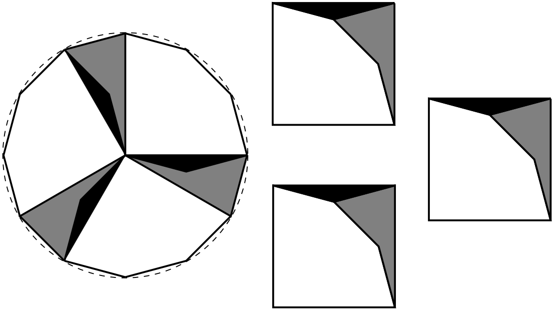


The area of a circle



- The area of a circle is clearly proportional to the *square* of its radius.
- That is, $A = kr^2$.
- Clearly, $k < 4$. Why?
- And $k > 2$. Why?

In fact, dissection of the regular dodecagon shows that $k > 3$.



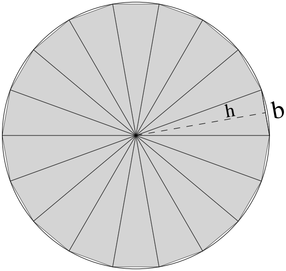
The area of a circle of radius r is $A = \pi r^2$

- In fact, $k = \pi$ (as you probably remember).
- Remember that π is *defined* in terms of *linear* measurements; it is the ratio of circumference to diameter.
- Thus, we have another deep relationship between length and area!

$$\frac{C}{2r} = \pi = \frac{A}{r^2}$$

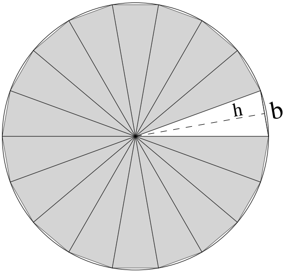
- Why does π, the ratio of circumference to diameter, also turn out to be the ratio of the *area* of the circle to the *area* of a square on the radius?
- Is it just a miracle, or can we understand the reason?

Why $A = \pi r^2$



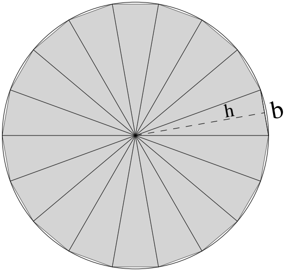
- As the number of sides, n, increases, the area of the inscribed n-gon approaches the area of the circle.

Why $A = \pi r^2$, continued



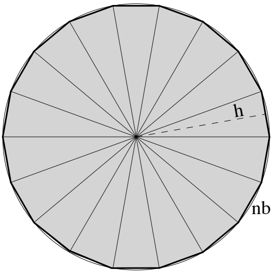
- Each triangle has area $\dfrac{1}{2}bh$.

Why $A = \pi r^2$, continued



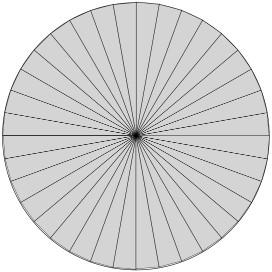
- So the area of the inscribed polygon is $\frac{n}{2}bh$. (There are n triangles.)

Why $A = \pi r^2$, continued



- nb is the perimeter of the polygon.
- As $n \to \infty$, $nb \to C$, the circumference of the circle, and $h \to r$, the radius of the circle. Remember that $C = 2\pi r$.

Why $A = \pi r^2$, conclusion



- Thus, as $n \to \infty$, the area of the inscribed polygon, $\dfrac{(nb)h}{2}$, approaches $\dfrac{2\pi r \cdot r}{2} = \pi r^2$.