MAT 2550, Introduction to Linear Algebra: Review Assignment

March 24, 2020

To be completed by class on Monday, March 30, for purposes of class discussion.

General note:

I have noticed that some of you are struggling with, and even resisting, the theory that underlies linear algebra. Theory and computation are complementary: theory provides a basis for understanding computational techniques, and computation illuminates understanding of the theory. To succeed in advanced mathematics, you have to do both. You cannot succeed by trying to memorize procedures; you have to think through them.

Concepts are established using definitions. Your first task is to understand these definitions. Theoretical results follow by arguing logically from these definitions, and you have to understand what they mean before you can logically develop a subject from them. I hope the following exercises will help you by giving concrete form to the essential concepts.

For each topic, a sample exercise is given, followed by an exercise, or a few exercises, for you to do on your own. This will give you way to test yourself. On Monday, March 30, at our first remote group meeting, we will discuss any problems with which you have had difficulty.

1. Deciding whether vectors are linearly independent or dependent.

   (a) (Example) Suppose vectors v_1, v_2, and v_3 are linearly independent. Show that $v_1 + v_2, v_2$, and v_3 are also linearly independent.

   Solution: Suppose $\alpha(v_1 + v_2) + \beta v_2 + \gamma v_3 = 0$. The goal is to show that $\alpha = \beta = \gamma = 0$. (Here we have introduced three arbitrary scalars, α, β, and γ. Notice that to avoid confusion I did not use subscripts for the scalar variables, since the first vector in the system, $v_1 + v_2$, involves two indices. It helps to choose good notation for the purpose at hand.) To achieve this goal, we will use the given information that v_1, v_2, and v_3 are linearly independent. Here goes!

   By the distributive property, we have

   $$\alpha v_1 + (\alpha + \beta)v_2 + \gamma v_3 = 0.$$

   By the linear independence of v_1, v_2, and v_3, we can assert that $\alpha = \alpha + \beta = \gamma = 0$. That gets us most of the way; we need only verify that $\beta = 0$. Since $\alpha = 0$ and $\alpha + \beta = 0$, we conclude by substitution and the identity property of 0 that $\beta = 0$ also.

   (b) (Example) Show that the polynomials 1, x, and $1 + x$ are linearly *dependent*.

   Solution: $1 + x + (-1)(1 + x) = 0$; thus, by defintion, these polynomials are linearly dependent.

(c) (Your turn!) Show that the vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \text{ and } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

are linearly independent.

(d) (Your turn again.) Suppose vectors v_1, v_2, and v_3 are linearly independent. Show that the vectors $2v_1$, $v_2 + 3v_3$, and $v_1 + v_3$ are linearly independent.

(e) (Your turn again.) Show that, given *any* vectors v_1, v_2, and v_3, the vectors $v_1 + v_2$, $v_1 + v_3$, and $v_3 - v_2$ are linearly dependent.

2. Understanding the span of a system of vectors.

(a) Show that the vector $\begin{bmatrix}1\\1\\1\end{bmatrix} \in \mathbb{R}^3$ is in the span of the vectors $\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\2\end{bmatrix}$.

Solution: We need to show that $\begin{bmatrix}1\\1\\1\end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination

$$\alpha \begin{bmatrix}2\\2\\1\end{bmatrix} + \beta \begin{bmatrix}1\\1\\2\end{bmatrix}.$$

By inspection it should be evident that we can take $\alpha = \beta = \frac{1}{3}$. (Note that this is the only way to do it, which reflect the fact that $\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\2\end{bmatrix}$ are linearly independent. Prove that they are!)

(b) (Your turn!) Show that $3 + x^2$ is in the span of $1 + 2x + x^2$ and $1 - x$.

(c) (Your turn again!) Show that $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \in \mathbb{R}^4$ is in the span of $\begin{bmatrix}2\\2\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\3\\4\end{bmatrix}$.

3. Finding a basis for a vector space.

(a) Extend the set $\left\{ \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix} \right\}$ to a basis for $\mathbb{R}^3$.

Solution: We know that the dimension of $\mathbb{R}^3$ is 3, and you should have proved above that the two given vectors are indeed linearly independent, a necessary condition for extending the set to a basis. To extend this set to a basis for $\mathbb{R}^3$, we need to find a vector that is *not* in the span $\left\langle \begin{bmatrix}2\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix} \right\rangle$. There are many possible answers; here is one, and I will invite you to share others. We can see that to get 0 in the first two coordinates of a linear combination, we have to make the third coordinate negative. So we can adduce the vector $\begin{bmatrix}0\\0\\1\end{bmatrix}$ to obtain a basis.

(b) You can check that the vectors $\begin{bmatrix}1\\2\end{bmatrix}$, $\begin{bmatrix}1\\1\end{bmatrix}$, and $\begin{bmatrix}2\\1\end{bmatrix}$ span $\mathbb{R}^2$; however, they cannot be linearly independent, since there are more vectors than the dimension of the space. Cut this set down to a basis for $\mathbb{R}^2$.

Solution: $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ is a basis. We can tell this is the case, since $\begin{bmatrix}1\\2\end{bmatrix} - \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$, and $\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$. We clearly span $\mathbb{R}^2$ with $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix}$. Since the dimension of $\mathbb{R}^2$ is 2, these vectors must be linearly independent.

Is the set $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$ a basis?

(c) (Your turn!) Verify that $\{1 + 2x + x^2, 1 - x\}$ is a linearly independent set and extend it to a basis for P_2.

(d) (Your turn again!) Extend your basis further to obtain a basis for P_3.

(e) (Your turn again) Cut the set $\{1 + x, 2 + x, 1 + 2x\}$ down to a basis for P_1.

4. Linear maps and matrices. Null space and range.

(a) Let $\{v_1, v_2, v_3\}$ be an ordered basis for a vector space V, let $\{w_1, w_2\}$ be an ordered basis for a vector space W, and let $\lambda : V \to W$ be the linear map defined by $\lambda v_1 = 2w_1 - w_2$, $\lambda v_2 = w_1 + w_2$, and $\lambda v_3 = w_1 + 2w_2$. (Assume the field of scalars is $\mathbb{R}$.)

i. What is $\lambda(v_1 - v_2 + 2v_3)$?

ii. What is the matrix for λ in terms of these bases?

iii. Find a basis for $\mathrm{Null}\lambda$.

iv. Find a basis for $\mathrm{Ran}\lambda$.

Solution:

i. $\lambda(v_1 - v_2 + 2v_3) = \lambda v_1 - \lambda v_2 + 2\lambda v_3 = (2w_1 - w_2) - (w_1 + w_2) + 2(w_1 + 2w_2) = 3w_1 + 2w_2$.

ii. Remember that the columns of the matrix give the outputs of v_1, v_2, and v_3 in terms of w_1 and w_2. (You should check that you understand why this is so.) So the matrix is

$$M_\lambda = \begin{bmatrix}2 & 1 & 1\\-1 & 1 & 2\end{bmatrix}.$$

iii. We need to identify the vectors $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ satisfying $\lambda v = 0$. These are, by definition, the vectors in $\mathrm{Null}\lambda$. We can use the matrix to do this:

$$\begin{bmatrix}2 & 1 & 1\\-1 & 1 & 2\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

Writing this equation in augmented matrix form and solving by row reduction, we obtain:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ -1 & 1 & 2 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 3 & 5 & 0 \end{array}\right]$$

$$\leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & \frac{5}{3} & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \end{array}\right].$$

So α_3 is free, $\alpha_2 = -\frac{5}{3}\alpha_3$, and $\alpha_1 = \frac{1}{3}\alpha_3$. A general solution now has the form

$$Null\lambda = \left\{\frac{1}{3}\alpha_3 v_1 - \frac{5}{3}\alpha_3 v_2 + \alpha_3 v_3 : \alpha_3 \in \mathbb{R}\right\} = \left\{\alpha\left(\frac{1}{3}v_1 - \frac{5}{3}v_2 + v_3\right) : \alpha \in \mathbb{R}\right\}.$$

Thus Nullλ is generated by the single vector $\frac{1}{3}v_1 - \frac{5}{3}v_2 + v_3$; in other words, that vector is a basis for Nullλ. (Note that the number of basis elements needed is always equal to the number of free variables in the solution. Note also that we could equally well choose $v_1 - 5v_2 + 3v_3$ as our basis element, for example.)

iv. Recall that Ranλ is the set of images λv. Since every $v \in V$ is a linear combination of v_1, v_2, and v_3, every image λv is a linear combination of $\lambda v_1 = 2w_1 - w_2$, $\lambda v_2 = w_1 + w_2$, and $\lambda v_3 = w_1 + 2w_2$. (That is what it means for λ to be linear.) Thus, $2w_1 - w_2$, $w_1 + w_2$, and $w_1 + 2w_2$ span Ranλ; however, they are not linearly independent, so we need to cut this set down. It is not hard to see that $\langle 2w_1 - w_2, w_1 + w_2, w_1 + 2w_2\rangle = W$, so we can just take $\{w_1, w_2\}$ as our basis. Of course, there are many other choices of basis you could make.

Both Nullλ and Ranλ are subspaces (of V and W, respectively), as you have seen. Make sure you know how to show that they are! In this case, dim Null$\lambda = 1$ and dim Ran$\lambda = 2$. Observe that dim Nullλ + dim Ranλ = dim V. This is not an accident!

(b) (Your turn!) Let $\{v_1, v_2, v_3, v_4\}$ be an ordered basis for a vector space V, let $\{w_1, w_2, w_3\}$ be an ordered basis for a vector space W, and let $\lambda : V \to W$ be the linear map defined by $\lambda v_1 = w_1 - w_2 + w_3$, $\lambda v_2 = w_1 - w_2$, $\lambda v_3 = w_3$, and $\lambda v_4 = w_3$. (Assume the field of scalars is $\mathbb{R}$.)

i. What is $\lambda(v_1 - v_2 + 2v_3 - v_4)$?

ii. What is the matrix for λ in terms of these bases?

iii. Find a basis for Nullλ.

iv. Find a basis for Ranλ.