MAT 2550: Final Exam **Name:** ______________________________

Due by 5 p.m. on Friday, May 8, 2020

You are expected to work on this exam alone and to refrain from talking about the exam questions to anyone except the professor until the time and date when it is due. You are welcome to help each other learn the relevant material, but not to discuss the specific problems on the exam. You may use your own notes and any published materials that you like. (Cite sources appropriately.)

Your signature constitutes a pledge that you have done the exam according to the above instructions. (Please sign your exam at the top of whatever paper you do it on.)

Signature:______________________________

1. Compute the following:

   (a) $2\begin{bmatrix}1\\0\end{bmatrix}+3\begin{bmatrix}0\\1\end{bmatrix}=$

   (b) $\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=$

   (c) $\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix}=$

2. Let $\lambda : V \to \mathbb{R}$ be a linear map. If $\lambda v_1 = 2$ and $\lambda v_2 = 3$, what is $\lambda(3v_1 + 2v_2)$?

3. Suppose v_1, v_2 is a basis for vector space V and w_1, w_2, w_3 is a basis for the vector space W. Provide the matrix, in terms of these bases, for the linear transformation $\lambda : V \to W$ defined by $\lambda v_1 = 3w_1 + 2w_2$ and $\lambda v_2 = w_1 + w_2 + w_3$.

4. Let $\lambda : V \to W$ be a linear map. Prove that $\text{Null}\lambda = \left\{ v \in V : \lambda v = \vec{0} \right\}$ is a subspace of V.

5. Consider the following three vectors: $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   Prove $\{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$.

6. Consider the linear map $\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ given (in terms of the standard basis $\{e_1, e_2, e_3\}$) by the matrix

$$M_\lambda = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Compute the determinant of M_λ.

(b) We know that λ is bijective, and hence a linear isomorphism, based on either Problem 5 or on the determinant you just calculated. Explain why. (Either explanation is acceptable.)

(c) Provide the matrix for the inverse to λ (in terms of the standard basis $\{e_1, e_2, e_3\}$).

7. Let $\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ be given by the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix}$$

(a) Provide a basis for Null λ.

(b) What is the column rank of this matrix? Justify your answer!

8. Let $\lambda : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by orthogonal projection onto the plane $y = 3x$.

(a) Provide the matrix for λ (in terms of the standard basis $\{e_1, e_2, e_3\}$).

(b) Is λ injective (but not surjective), surjective (but not injective), bijective, or neither? (Circle one, or write one on your paper.)

Injective (only) Surjective (only) Bijective Neither

(c) What is the dimension of Null λ?

(d) What is the dimension of Ran λ?

9. Consider the following matrix M:

$$M = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of M.

(b) Provide a basis of eigenvectors for M. Order it with the eigenvector for the smaller eigenvalue first. Choose the eigenvectors with whole number coordinates such that the first coordinate is positive and as small as possible.

10. Consider again the matrix $M = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ of the previous problem.

(a) Provide the change-of-basis matrix that writes your basis of eigenvectors in terms of the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$

(b) Provide the inverse change-of-basis matrix that changes from the standard basis back to your basis of eigenvectors.

(c) Conjugate matrix M by these change-of-basis matrices (that is, multiply by the appropriate change-of-basis matrix on left and right) and verify by multiplying the three matrices that the result is the diagonal matrix of eigenvalues.