MAT 2550: Inverse Matrices and Change of Basis Matrices

April 14, 2020

To be handed in. Due by 5 p.m. on Wednesday, April 22, 2020.

In each of the problems below, an invertible matrix is given. (You can tell it is invertible, because the matrix is square and the rows and columns will be readily seen to be independent. In any case, if you successfully calculate an inverse matrix by reducing the matrix on the left to the identity, you will know the matrix is invertible. For any matrix that is not invertible, this process will fail.) Calculate the inverse matrix. Please also check your answer by multiplying the matrices.

Remark. Each $n \times n$ matrix would represent a linear map between vector spaces of the same dimension, possibly between a vector space and itself (in which case the map is called a *linear operator*), once an n-dimensional domain and range are chosen, along with bases for these spaces. The inverse matrix would then represent the inverse map in terms of the same bases (with domain and range exchanged, of course). We might most naturally think of these matrices as representing maps from $\mathbb{R}^n$ to itself, in terms of the standard basis of column vectors – and that is of course how we calculate with them, although in applications these standard basis vectors may be identified with an ordered basis for a vector space described in another way, such as a space of polynomials. In summary, when doing these problems, you need not worry about a choice of basis; they are purely problems in matrix algebra.

Example. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \leftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

The inverse matrix is $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

Check: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. We need only check the product in one order, since we know by associativity of matrix multiplication that if the inverse matrix works on one side, it *has* to work on the other side. (See the proposition on the last page of the *Properties of Functions* worksheet.)

1. $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 4 \end{bmatrix}$
4. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
5. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

As you have probably noticed, matrices are like chameleons: each matrix can represent many different linear maps, depending on the vector spaces and ordered bases chosen. Here are two ways to look at our sample matrix, $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$, as a map from $\mathbb{R}^3$ to itself. On the one hand, in terms of the standard basis for both the domain and range, it represents the linear map λ defined by $\lambda\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\lambda\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$, and $\lambda\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. On the other hand, if we choose the ordered basis $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ for the domain and the standard basis for the range, it is the identity map: it takes each of these vectors to itself, but in terms of a different basis. With this second interpretation, it is called a *change-of-basis* matrix.

Since writing column vectors in terms of a different basis of column vectors is a bit confusing, this might be easier to understand if we instead think of our matrix as representing a map from P_2 to itself. In terms of the basis $\{1, 1+x, 1+x+x^2\}$, we make the identification

$$1 \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad 1+x \leftrightarrow \begin{bmatrix}0\\1\\0\end{bmatrix}, \qquad 1+x+x^2 \leftrightarrow \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

In terms of the standard basis $\{1, x, x^2\}$, we make the identification

$$1 \leftrightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}, \qquad 1+x \leftrightarrow \begin{bmatrix}1\\1\\0\end{bmatrix}, \qquad 1+x+x^2 \leftrightarrow \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

Multiplying by the matrix $\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$ makes the translation, writing the same vectors in terms of the standard basis:

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

O.K., maybe that is not very exciting, since it is obvious how to do that! The cool part is that the inverse matrix makes the translation the other way, the way that is harder to see:

$$\begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\end{bmatrix},$$

telling us immediately that $1 = 1(1) + 0(1+x) + 0(1+x+x^2)$, $x = (-1)(1) + 1(1+x) + 0(1+x+x^2)$, and $x^2 = 0(1) + (-1)(1+x) + 1(1+x+x^2)$. That was not so hard to see by inspection, either, but in more difficult cases inverse change-of-basis matrices are extremely useful. (And you need them to program computers, even in the easy cases; computers cannot solve equations by inspection!)

If we let $B = \{1, x, x^2\}$ and $B' = \{1, 1 + x, 1 + x + x^2\}$, we can write

$$M_{B' \mapsto B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad M_{B \to B'} = M_{B' \mapsto B}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

In each of the following exercises, provide $M_{B' \mapsto B}$ and $M_{B \mapsto B'}$, where $B = \{1, x, x^2\}$. (I said "provide" rather than "compute" because you have already done all of the computational work! Look at the previous problems!) Remember that the order of the basis matters.

6. $B' = \{1 + x^2, 2 + 2x + 2x^2, 1 + 2x + 3x^2\}$
7. $B' = \{1 + x + x^2, 1 + x, 2 + 3x + 2x^2\}$
8. $B' = \{1 + x + x^2, 1 + 2x + 2x^2, 1 + 2x + 4x^2\}$
9. $B' = \{1 + x + x^2, 1 + x, 1\}$
10. $B' = \{1 + x + 2x^2, 1 + 2x + 2x^2, 1 + 2x + x^2\}$