MAT 2550, Introduction to Linear Algebra: Initial Assignment

January 5, 2019

To be completed by the beginning of class on Wednesday, January 9.

1 Linear Equations

I expect you are familiar with systems of linear equations. For example, they come up when using the partial fractions method for separating the integral of a rational function. Here is an example:

$$\frac{A}{x+1} + \frac{B}{2-x} = \frac{x+7}{(x+1)(2-x)}$$

We wish to find A and B. Putting both fractions on the left in terms of the common denominator, we obtain

$$\frac{A(2-x)+B(x+1)}{(x+1)(2-x)} = \frac{x+7}{(x+1)(2-x)},$$

which clearly is true if and only if

$$A(2-x)+B(x+1) = x+7.$$

At this point, substituting $x = 2$ and $x = -1$ provides a shortcut to the solution, but let us leave that aside, because we seek a method that will work more generally, when this trick does not apply. Our strategy is to use this simple example as a basis for understanding far more general concepts.

Combining like terms, we obtain

$$(B-A)x + (2A+B) = x+7.$$

This equation will be true for all values of x if and only if the following *system* of two *linear* equations are true:

$$B - A = 1$$
$$2A + B = 7.$$

At this point, you should once again see a shortcut: substitute $B = A + 1$, which is justified by the first equation, into the second equation, obtaining $3A + 1 = 7$. Once again, we will leave this shortcut aside in search of deeper understanding. Note in particular that if the system had three or more variables, this substitution would generally not work. (A good way to understand math is to analyze simple examples without using anything special about them.) However, let us note at this point that $A = 2$ and $B = 3$ is the unique solution: B *must* exceed A by 1 – the first equation says so, and thus $3A$ *must* equal 6 (subtracting 1 from both sides of the second, after substituting); conversely, these values for A and B do make the equations true. Also, now you can forget about the partial fractions problem from which this system of equations arose; our aim is just to understand how to solve such systems of equations – including those with many more variables and equations – and understand the more general mathematical context they represent.

First, note that we have an *ordered pair* of variables, (A, B). We can think of our solution as the corresponding ordered pair $(2, 3)$. The order of the variables matters. We need to know which is A and which is B, since setting $A = 3$ and $B = 2$ does not give a solution, and choosing an order for the variables allows us to keep track of that. (We could just as well have ordered them (B, A), in which case our solution would be the ordered pair $(3, 2)$, but alphabetical order is more natural. All that matters is to be consistent.) Next, let's focus on the left hand sides, ignoring the right hand side for a moment. These are expressions in the two variables A and B, and they are *linear* in the sense that the variables each occur to the first power. (Nothing squared, cubed, square-rooted, etc.) Expressions tell us to do something with the variables, and so they define *functions*. We have two functions of the ordered pair (A, B): $f(A, B) = B - A$, and $g(A, B) = 2A + B$. Recall that A and B are the inputs, $f(A, B)$ and $g(A, B)$ are the outputs, and the functions f and g specify (with formulas, in this case) how the inputs and outputs are *related*. The *domain* of the inputs can be any algebraic system in which we can perform the required operations: addition, subtraction, and multiplication. In our example, the natural domain for the problem is the real number system.

Let us combine the outputs into an ordered pair, obtaining a single function, which we'll denote for the moment as (f, g), with a pair of inputs (that is, a function of two variables) and a pair of outputs:

$$(f, g)(A, B) = (f(A, B), g(A, B)) = (B - A, 2A + B).$$

The function (f, g) is called a *linear function*, since each of its outputs is given by a linear expression. One way to think about a linear function is that each output is proportional to each input (if the other inputs are set to zero), and the contributions of the inputs are additive. Linear functions are more typically called linear *maps*, for reasons that will become apparent when we study their geometric representations. We will also see later that there is also another, very important way to characterize linear maps. For simplicity, let us rename our function λ, for linear, since λ is the Greek letter corresponding to l in our alphabet. Thus,

$$\lambda(A, B) = (B - A, 2A + B).$$

It is always helpful to have nice ways to write things, and there is a very nice way to write the formula for a linear function. Think of the pair (A, B) as a column and the coefficients for each output as a row, making sure to keep the order, so $B - A$ becomes $-A + B = (-1)A + 1B$, obtaining a *matrix* formula:

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} (-1)A + 1B \\ 2A + 1B \end{pmatrix} = \begin{pmatrix} B - A \\ 2A + B \end{pmatrix}$$

I hope you can see from this example how matrix multiplication is done; it's easier to see from an example than to describe in words.

Now we can see that solving the system of linear equations is equivalent to finding which input pair gives the specified output pair $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ when the linear function λ is applied:

$$\lambda \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

We have translated our problem into the familiar and powerful language of functions, inputs, and outputs. We will develop a systematic method for determining the set of solutions to a functional equation of this type. Our first task, however, is to become competent in doing matrix computations. Competence is attained through practice, that is, through exercise. Here (on the following page) are some exercises; if you need more practice, you can easily make up your own.

2 Exercises

Perform each of the following matrix calculations, as demonstrated by the example above and the examples below.

Example: $$\begin{pmatrix} -2 & 1 & 3 \\ 2 & -3 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (-2)x + y + 3z \\ 2x - 3y + \frac{2}{3}z \end{pmatrix}$$

Example: $$\begin{pmatrix} -2 & 1 & 3 \\ 2 & -3 & \frac{2}{3} \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (-2)x + y + 3z \\ 2x - 3y + \frac{2}{3}z \\ x - z \end{pmatrix}$$

Example: $$\begin{pmatrix} -2 & 1 \\ 2 & -3 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (-2)x + y \\ 2x - 3y \\ y \\ x \end{pmatrix}$$

Example: $$\begin{pmatrix} -2 & 1 \\ 2 & -3 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2)2 + 1 \\ 2(2) - 3(1) \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

1. $$\begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

2. $$\begin{pmatrix} -2 & 1 \\ 0 & -3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =$$

3. $$\begin{pmatrix} -2 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} =$$

4. $$\begin{pmatrix} -2 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} =$$

5. $$\begin{pmatrix} -2 & 1 & \frac{2}{3} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} =$$

Explain why the following expressions do not make sense, and the calculations cannot be performed.

$$\begin{pmatrix} -2 & 1 & 0 & -3 \\ 0 & -3 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} -2 & 1 \\ 0 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$