MAT 2443: Exam 2 **Name:** ______________________________

March 19, 2014.

No calculators, notes, or books are allowed. You may have only writing implements (including a ruler or other drawing aids) and blank paper.

Each numbered question is worth 20 points; any lettered parts of a question have the same value.

1. *Parametrized Curves.* A particle travels along the curve in space defined by $\mathbf{r}_1(t) = (\cos t, \sin t, t)$, where position is measured in meters and t represents time, in seconds (after some initial instant). A second particle travels along the vertical straight line defined by $\mathbf{r}_2(t) = (0, 1, 2t)$.

   (a) It should be obvious to you that these particles do not collide. Explain! (Hint: Look at their third coordinates.)

   (b) It is known that there is a unique solution to the equation $t = \cos t$, which lies on the interval $(0, 1)$; call this solution α. Show that the particles are nearest to each other when $t = \alpha$ seconds.

2. (a) *Differentiation.* Let $F : \mathbb{R} \to \mathbb{R}^3$ and $G : \mathbb{R} \to \mathbb{R}^3$ be differentiable functions, and suppose that $F(0) = (1, 0, 0)$, $F'(0) = (1, 1, 1)$, $G(0) = (0, 1, 0)$, and $G'(0) = (1, 1, 0)$. Evaluate $(F \cdot G)'(0)$ (where the function $F \cdot G$ is defined in the obvious way by $F \cdot G(t) = F(t) \cdot G(t)$).

   (b) *Parametrization and arc length.* Write down an integral expression for the circumference (that is, arc length once around) of the ellipse in which the cylinder $x^2 + y^2 = 1$ intersects the plane $z = x + y$. *You need not evaluate this integral.*

3. *Velocity, speed, orthonormal frame, curvature, & acceleration.* A particle moves along the helical path $\mathbf{r}(t) = (\cos t, \sin t, t)$, where distance is measured in meters and t is time in seconds.

   (a) Calculate the speed of the particle, showing it is constant.

(b) Calculate the unit tangent, normal, and binormal vectors, $\mathbf{T}(t), \mathbf{N}(t)$, and $\mathbf{B}(t)$.

(c) Calculate the curvature of the path of the particle, showing that it is constant.

(d) Let $\mathbf{a}(t) = a_T\mathbf{T} + a_N\mathbf{N}$ be the acceleration of the particle, where a_T and a_N are its tangential and normal (scalar) components, respectively. Show that $a_T = 0$ – this should make sense to you, given that the speed is constant, and calculate a_N, which is constant, but not zero – this should make sense to you, too, since both speed and curvature are constant.

4. *Level curves & limits.* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = e^{x^2+y^2} - 1$.

   (a) Show that the level curves of f are circles centered at the origin.

   (b) Evaluate $\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{x^2+y^2}$.

5. *Partial derivatives.* Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) = e^{xy}$.

   (a) Calculate $\frac{\partial}{\partial x} f(0, 0)$.

   (b) Calculate $\frac{\partial^2}{\partial y \partial x} f(0, 0)$.

Extra Credit! (5 points) *Solutions due by the beginning of class on Monday, March 24.*

Prove that there is a solution to the equation $t = \cos t$ on the interval $(0, 1)$ and that there is no other solution to this equation, a fact that was used in Problem 1. Illustrate your argument using the graphs of the functions $f(t) = t$ and $f(t) = \cos t$.

Hints: To prove there is a solution on the given interval, use the Intermediate Value Theorem. To show this solution is unique, observe that if $t > 1$, then $t - \cos t > 0$, since $\cos t \leq 1$. Make a similar argument for $t < -1$. Finally, observe that for $-1 \leq t < 0$, $\cos t$ and t have opposite signs.