MAT 2442: Exercises on Power Series, Particularly Taylor and Maclaurin Series

Taylor and Maclaurin Series are essential mathematical tools. Understanding these power series makes use of all the aspects of sequences and series we have studied. Some emphasis will be placed on putting correct bounds on the possible error made in approximating the values of functions and definite integrals using Taylor and Maclaurin series.

Recall that Taylor's Theorem gives a nice but somewhat indeterminate formula for the remainder, $R_n(x)$, when $f(x)$ is approximated by its n^{th} degree Taylor polynomial. Taylor's formula,

$$R_n(x) = \frac{f^{(n+1)}(z)(x-a)^{n+1}}{(n+1)!},$$

where a is the center of the Taylor series and z is between a and x, is a natural generalization of the Mean Value Theorem. (It's proof is elementary but somewhat involved. A Maclaurin Series is simply the special case that $a = 0$.) Observe that the remainders $R_n(x)$ define a *sequence*, and the Taylor series converges to $f(x)$ if and only if the limit of this sequence as $n \to \infty$ is zero. The following three problems deal with this issue.

1. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

2. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$.

3. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3n}$.

4. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 3^n}$.

5. Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 3^n}$.

6. Explain why $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for any value of x. (It is not sufficient just to say that $n!$ grows faster; explain the pattern that emerges as n grows. It might help to start with a simple example, such as $x = 2$.)

7. Since $f(x) = e^x$ is an increasing function, it is clear that for $0 \le z \le x$, $e^z \le e^x$. Use this fact and Taylor's formula to show that the Maclaurin series for e^x converges to e^x for every positive value of x.

8. It should also be clear that for $x \le z \le 0$, $e^z \le 1$. Use this fact and Taylor's formula to show that the Taylor series for e^x converges to e^x for every negative value of x.

   Remark: It should be obvious that a Maclaurin series converges at $x = 0$.

9. Use Taylor's Theorem to prove that $\sin x = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for any real value of x. (Hint: $|\sin x| \leq 1$ and $|\cos x| \leq 1$ for all values of x.)
10. Take the derivative of the series above to derive the Maclaurin series for $\cos x$.
11. Find the MacLaurin series for $f(x) = x^2 \cos x$.
12. Find the MacLaurin series for $f(x) = \cos(x^2)$.
13. Use long division to calculate the third-degree Maclaurin polynomial (that is, Taylor polynomial centered at $a = 0$) for $\tan x$.
14. Use multiplication of series to calculate the third degree Maclaurin polynomial for $e^x \cos x$. [Hint: For the term of each degree, look for the terms in each factor that multiply to that degree as you go along.]
15. Derive the Maclaurin series for $\sqrt{1+x}$ and use it to calculate $\sqrt{1.1}$ with error at most .01. (You need not find the radius on which the series converges to $\sqrt{1+x}$, but if you'd like a bit of a challenge, see if you can!)
16. Derive the Maclaurin series for $\sqrt{4+x}$ and use it to calculate $\sqrt{5}$ with error at most .01.
17. Derive the Maclaurin series for $\frac{1}{\sqrt{1+x}}$.
18. Approximate $\sqrt[3]{e}$ with possible error less than .005.
19. Approximate $\frac{1}{\sqrt[3]{e}}$ with possible error less than .005. (You don't need Taylor's Theorem for this one, because the series alternates, but it is worth bounding the error using both methods for practice; you will get the same answer.)
20. Use the MacLaurin Series for $\ln(1+x)$, evaluated at $x = -\frac{1}{2}$, to calculate $\ln 2$ with possible error less than .01. [Hint: We discussed this briefly in class on Tuesday. Observe that $\ln(2) = -\ln\left(\frac{1}{2}\right)$ and that, for any partial sum, the remainder of the series for $\ln(2)$ obtained in this way is bounded above by a convergent geometric series. You won't need very many terms.]
21. Derive the MacLaurin series for $\arctan x$ and determine its radius of convergence. (Hint: You don't need to use the formula $c_n = \frac{f^{(n)}(0)}{n!}$ or Taylor's Remainder Formula, although you certainly can if you want to and might benefit from the practice. There is an easier way: integrate a geometric series. If $f(x) = \arctan x$, what is $f'(x)$?)
22. Approximate $\displaystyle\int_0^{.1} \sin(\theta^4)d\theta$ with possible error less than 10^{-13}.
23. Suppose $f(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n^2}$, $-1 \leq x \leq 1$. What is $f^{(3)}(0)$?
24. Suppose $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function. Suppose that $f(0) = 0$, $f'(0) = 1$, $f^{(2)}(0) = 2$, and $f^{(3)}(0) = 3$. What is the 3^{rd}-degree MacLaurin polynomial for f?
25. More generally, suppose $f^{(n)}(0) = n$ (following the pattern above). Suppose in addition that $|f^{(n)}(x)| \leq n$ for every real number x. Provide the MacLaurin series for f and prove that it converges to f for all real numbers.