Review of Integration & the Fundamental Theorem of Calculus

Introduction to the Natural Logarithm

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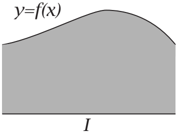
Incremental Accumulation of Oriented Quantities

- Integration is a means of understanding and computing the incremental accumulation of oriented quantities such as:
  - displacement
  - velocity
  - work
- Although it might surprise you at first, area and volume are also best understood as oriented quantities.
- To see this, it might help to visualize volume as liquid in a tank: it can go up or down.

Canonical Example: The Oriented Area Bounded by the Graph of a Function

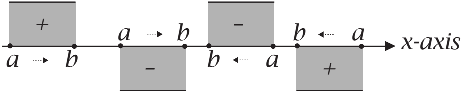
- As another example, let f be continuous on an interval.
- It seems evident that the region bounded by the graph of f and the x-axis along this interval has a well-defined area.



- To see this area as a quantity that *accumulates* with changes in x, we must view it an *oriented* quantity: just as with displacement, any accumulation can be reversed.
- We will now justify that this region has a well-defined *oriented* area. Then we will see how to compute it with ease for many functions by first understanding its derivative.

The Oriented Area Bounded by the Graph of a Constant Function

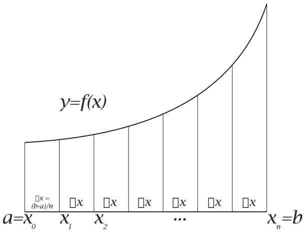
- First consider a constant function $f(x) = c$. The oriented area bounded by the graph of f *from a to b* is $(b - a)c$. Note that this quantity is:
  - 0 if $a = b$ or $c = 0$.
  - Positive if $a < b$ and $c > 0$
  - Negative if $a < b$ and $c < 0$
  - Negative if $a > b$ and $c > 0$
  - Positive if $a > b$ and $c < 0$



- The magnitude of the oriented area is to oriented area as distance is to displacement and as speed is to velocity.

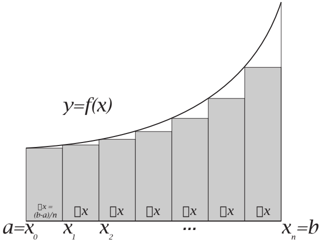
The Oriented Area Bounded by the Graph of a Function

- Now consider any function that is continuous on an interval, along with two points a and b in that interval.
- Let us focus first on the case that $a < b$ and f is positive and increasing on $[a, b]$.
- For each positive integer n, consider the *partition* of $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. (Δx clearly depends on n, but it is cumbersome to incorporate n into the notation.)

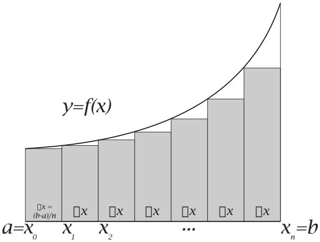


The Region Bounded by the n^{th} Lower Piece-wise Constant Approximation

- On each interval $[x_i, x_{i+1}]$, consider the minimum value of f; in this case, it will be $f(x_i)$.
- The oriented area bounded by the graph of the constant function $g_i(x) = f(x_i)$ is $f(x_i)\Delta x$.
- The region bounded from a to b by the graph of the piecewise constant function $g(x) = g_i(x)$ for $x \in [x_i, x_{i+1}]$ is contained in the region bounded by the graph of f.



The Oriented Area bounded by the n^{th} Lower Piece-wise Constant Approximation



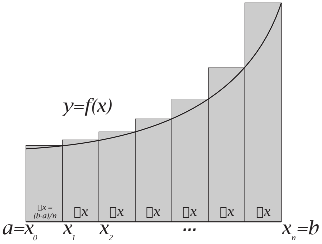
Its oriented area is $L_n =$

$$f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x$$

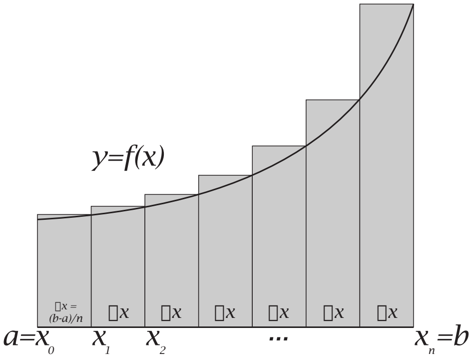
$$= \sum_{i=0}^{n-1} f(x_i)\Delta x.$$

The Region Bounded by the n^{th} Upper Piece-wise Constant Approximation

- On each interval $[x_i, x_{i+1}]$, consider the maximum value of f; in this case, it will be $f(x_{i+1})$.
- The oriented area bounded by the graph of the constant function $h_i(x) = f(x_{i+1})$ is $f(x_{i+1})\Delta x$.
- The region bounded from a to b by the graph of the piecewise constant function $h(x) = h_i(x)$ for $x \in [x_i, x_{i+1}]$ clearly contains the region bounded by the graph of f.

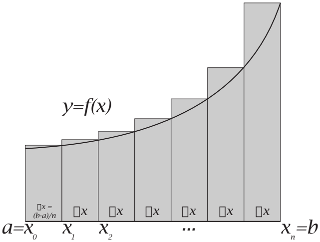


The Oriented Area Bounded by the n^{th} Upper Piece-wise Constant Approximation



Its oriented area is $U_n =$?

The Oriented Area Bounded by the n^{th} Upper Piece-wise Constant Approximation



Its oriented area is $U_n =$

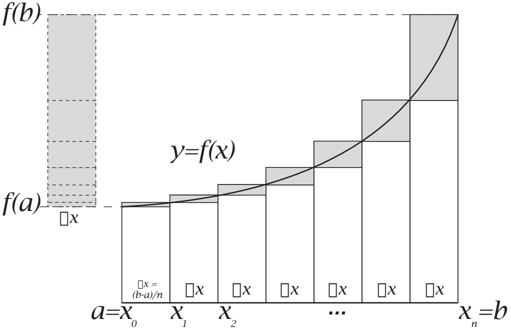
$$f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{i=1}^{n} f(x_i)\Delta x.$$

The Limits as $n \to \infty$

- As we let $n \to \infty$, L_n increases and U_n decreases. (Why?)
- Clearly $L_m \leq U_n$ for any m and n.
- By the continuity of the real number system, $\lim_{n\to\infty} L_n$ and $\lim_{n\to\infty} U_n$ must exist.
- In fact, these limits are the same, as we will see in a moment.
- Since the region bounded by the function f contains the region bounded by each lower piece-wise constant approximation and is contained in the region bounded by each upper piece-wise constant approximation, its area must be this common limit.

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n =$ The Oriented Area of the Region bounded by f

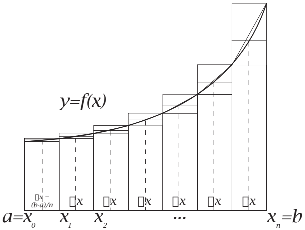


- The difference between L_n and U_n, in this case, is $(f(b) - f(a))\Delta x_n$.
- As $n \to \infty$, $\Delta x_n \to 0$; hence this difference goes to 0.

Oriented Area in General

- A similar argument shows this is the case on intervals on which f is decreasing.
- It also causes no difficulty if $f(x) \leq 0$ for some inputs x or if $b < a$. Some of the approximating constant pieces may bound negative areas, but all of the relationships and reasoning extend to the general case.
- There is also no need to subdivide the interval from a to b into equal subintervals; any subintervals will do as long as their maximum length goes to 0 as $n \to \infty$.
- For less tractable – but continuous – functions, such as $f(x) = x \sin(\frac{\pi}{x})$ on an interval containing 0, the domain cannot be divided into finitely many intervals of increase or decrease. For such functions, more refined arguments are needed. Nonetheless, all of the results above hold.

Other Approximations



- Furthermore, all other approximations, such as the midpoint and trapezoidal approximations, are squeezed between L_n and U_n as well.
- These other approximations (especially the trapezoidal) converge more quickly than L_n and U_n.
- But we can often calculate the area exactly.

Extension of these Methods to Other Measures; Properties of Measure

- The methods we used to define the oriented area bounded by a curve extend naturally to other measures such as the oriented volume bounded by a surface.
- They also extend more generally to physical quantities such as work.

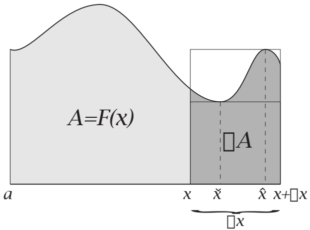
Oriented Area as a Variable Quantity

- Consider the oriented area bounded by a continuous function f from an initial input a to x.
- This oriented area is a function of x: call it $A = F(x)$.
- We will first compute $F'(x)$ and use it to compute $F(x)$.
- Note that to do this we must view the oriented area bounded by f as a quantity that *varies* with the ending point of the interval, x, just as we viewed velocity as a quantity that varied with time.
- Recall that

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}.$$

Bounding the Derivative of the Oriented Area

- Let $f(\check{x})$ be the minimum value of f on the interval form x to $x + \Delta x$, and let $f(\hat{x})$ be the maximum value of f on the interval form x to $x + \Delta x$.



- Then $f(\check{x})\Delta x \leq \Delta A \leq f(\hat{x})\Delta x$; hence,

$$f(\check{x}) \leq \frac{\Delta A}{\Delta x} \leq f(\hat{x}).$$

Computing the Derivative of the Oriented Area Using the Squeeze Theorem

- Since $\check{x}$ and $\hat{x}$ are between x and $x + \Delta x$, $\check{x} \to x$ and $\hat{x} \to x$ as $\Delta x \to 0$.
- Since f is continuous, $\lim_{\check{x}\to x} f(\check{x}) = \lim_{\hat{x}\to x} f(\hat{x}) = f(x)$.
- Combining the two previous observations, we obtain

$$\lim_{\Delta x\to 0} f(\check{x}) = \lim_{\Delta x\to 0} f(\hat{x}) = f(x).$$

- Thus, by the Squeeze Theorem,

$$F'(x) = \lim_{\Delta x\to 0} \frac{\Delta A}{\Delta x} = f(x)!$$

- The fact that $F' = f$ and the initial condition $F(a) = 0$ completely determine the function F.

The Riemann Integral: Formal Definition

The general concept that captures all of these examples and more is the Riemann Integral:

Definition

Let f be a function defined on the interval from a to b. For each positive integer n, let $\Delta x = \frac{b-a}{n}$. Let x_i^* be any point between x_{i-1} and x_i The *Riemann integral* of f from a to b, denoted by $\int_a^b f(x)dx$, is the following limit, if it exists and has the same value for all choices of x_i^*:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

Remark: The interval from a to b is *directed*: a can be greater than b; Δx can be negative.

Existence of the Riemann Integral

- If the Riemann integral of f from a to b exists, then f is *Riemann integrable* on the interval from a to b.
- A sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an n^{th} Riemann Sum.
- If $f(\check{x}_i)$ is the minimum value of $f(x)$ between x_{i-1} and x_i, then $\sum_{i=1}^{n} f(\check{x}_i)\Delta x$ is the n^{th} lower Riemann Sum.
- If $f(\hat{x}_i)$ is the maximum value of $f(x)$ between x_{i-1} and x_i, then $\sum_{i=1}^{n} f(\hat{x}_i)\Delta x$ is the n^{th} upper Riemann Sum.
- For a function to be Riemann integrable on an interval, it suffices that the upper and lower Riemann sums converge to the same limit as $n \to \infty$.
- Continuity of f on an interval is sufficient to guarantee that f is Riemann integrable on this interval. (Weaker but more complicated conditions also suffice.)

Fundamental Properties of The Riemann Integral

- The value of the Riemann integral is clearly the oriented area bounded by the graph of f from a to b. But it provides a general concept and notation that is valuable in situations where we don't necessarily want to picture this quantity as an oriented area.
- The following important properties of the Riemann integral are consequences of its definition:
  1. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$. ($\int_a^a f(x)dx = 0$.)
  2. $\int_b^a f(x)dx = -\int_a^b f(x)dx$.
  3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.
  4. For real numbers λ and μ, $\int_a^b [\lambda f(x) + \mu g(x)]dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$.
- Make sure you can explain them using pictures!

The Fundamental Theorem of Calculus

- Using the properties just mentioned and translating our reasoning about area into the modern language of integration, we state our most significant theorem:

Theorem (The Fundamental Theorem of Calculus)

Let f be a function that is continuous on $[a, b]$.

1. *Let $F(x) = \int_a^x f(t)dt$, $a \leq x \leq b$. Then F is continuous on $[a, b]$ and differentiable on (a, b), and $F'(x) = f(x)$.*
2. *If G be any anti-derivative for f, then* $\int_a^b f(x)dx = G(b) - G(a)$.

- Please take note of the hypothesis: the result does *not* apply *unless* the function f is *continuous* on $[a, b]$.

Proof of the Fundamental Theorem of Calculus

1. $F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x)-F(x)}{\Delta x}$, by the definition of the derivative, and $\frac{F(x+\Delta x)-F(x)}{\Delta x} = \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$, by the definition of the function F.

$$\frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} = \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}$$

by Property (1) of the Riemann integral, as listed previously. Since f is continuous on $[x, x+\Delta x]$, the Extreme Value Theorem applies, giving points $\check{x}$ and $\hat{x}$ such that $f(\check{x})$ and $f(\hat{x})$ are absolute minimum and maximum values on $[x, x+\Delta x]$, respectively. By Property (2) of the Riemann integral, as listed previously,

$$f(\check{x})\Delta x \le \int_x^{x+\Delta x} f(t)dt \le f(\hat{x})\Delta x$$

Proof of the Fundamental Theorem, Continued

Hence, $f(\check{x}) \leq \dfrac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} \leq f(\hat{x})$.

As $\Delta x \to 0$, clearly $\check{x} \to x$ and $\hat{x} \to x$ (since $x \leq \check{x}, \hat{x} \leq x + \Delta x$); furthermore, since f is continuous, as $\check{x} \to x$ and $\hat{x} \to x$, $f(\check{x}) \to f(x)$ and $f(\hat{x}) \to f(x)$ (by definition of continuity). Thus, by the Squeeze Theorem,

$$F'(x) = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x} = f(x)$$

2. By definition of F, $\int_a^b f(x)dx = F(b)$. Since $G' = F'$, $F = G + C$, where C is a constant. Since $F(a) = 0$, $C = -G(a)$. Thus $\int_a^b f(x)dx = F(b) = G(b) - G(a)$.

$\mathcal{Q.E.D.}$

An Important Function: The Anti-Derivative of $f(x) = \frac{1}{x}$

- Consider the function $f(x) = \frac{1}{x}$ for $x > 0$. (On the interval $(0, \infty)$, f is continuous.)
- Is there a function F such that $F'(x) = f(x) = \frac{1}{x}$?
- Yes, of course! It is given by $F(x) = \displaystyle\int_1^x \frac{1}{t} dt$, the oriented area under the curve $y = \frac{1}{t}$ between $t = 1$ and $t = x$.
  - We choose 1 for the starting point because we want $F(1) = 0$ (for reasons that will become apparent shortly).
  - Any other fixed positive number a would do, and would simply give a function that differs from ours by a constant.
  - That constant would be $\int_a^1 \frac{1}{t} dt$, the oriented area under the curve between $x = a$ and $x = 1$.
  - Since $y > 0$, $F(x) = \int_1^x \frac{1}{t} dt$ is positive if $x > 1$ and negative for $x < 1$.

Calculating $F(x) = \int_1^x \frac{1}{t} dt$.

- A harder question: Is there a formula for F constructed by adding, subtracting, multiplying, dividing, or composing algebraic and trigonometric functions?
- No!
- But we can calculate the values of $F(x)$, for any positive input x, to any desired degree of accuracy using Riemann Sums to estimate the Riemann Integral.
- First let us see why the function F is so important!

We now turn to Assignment 1, Section 1.

The Function F is the *Natural Logarithm*

- The *transcendental* function F is important enough to have a name, just like the trigonometric functions.
- F is called the *natural logarithm*, abbreviated ln. Why?
- Consider: by definition, $\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt =$

$$\int_1^a \frac{1}{t}dt + \int_a^{ab} \frac{1}{t}dt = \ln a + \int_a^{ab} \frac{1}{t}dt.$$

- For the second integral, substitute $u = \frac{t}{a}$, obtaining

$$\ln(ab) = \ln(a) + \ln(b)$$

- F is a logarithmic function! It follows that for any rational number r, $\ln(a^r) = r\ln a$. (Why?)

The Natural Exponential

- The natural logarithm is differentiable and increasing, hence injective, on the interval $(0, \infty)$.
- The natural logarithm is surjective onto $\mathbb{R}$. Why?
- Thus ln has a differentiable inverse $\exp : \mathbb{R} \to (0, \infty)$. On the same gid, graph $y = \ln x$ and $y = \exp x$.
- The inverse of a logarithmic function is an exponential function: $\exp(x + y) = \exp(x) \cdot \exp(y)$. Why? Hint: $x = \ln u$ and $y = \ln v$ for some real numbers u and v.
- Thus for any rational number r, $\exp(rx) = \exp(x)^r$. Why?
- Let $e = \exp(1)$. Then for any rational number r, $\exp(r) = e^r$. Why?
- More generally, since e^r agrees with $\exp(r)$ for any *rational* number r, it is natural to define $e^x = \exp(x)$ for any *real* number x (whether rational or irrational).

Now let's do some exercises to clarify our understanding of the exponential function!

Estimating the value of e

- Using a little trick with the derivative of the natural logarithm, along with the continuity and computational properties of the exponential function, we can express the number e as limit:

$$e = e^1 = e^{\ln'(1)} = e^{\lim_{h\to 0} \frac{\ln(1+h)-\ln(1)}{h}} = \lim_{h\to 0} e^{\frac{1}{h}\ln(1+h)}$$

$$= \lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

- Using this limit we can estimate e to any degree of accuracy that we wish. Let's do it!

The Derivative of the Natural Exponential Function

- To calculate the derivative of the exponential function, we use implicit differentiation, which works whenever we know the derivative of the inverse of a function.
  - Observe that $\ln(e^x) = x$. Let
  - Let $u = e^x$. Taking the derivative of each side, applying the chain rule to the composition, we obtain $\frac{1}{u} \cdot \frac{du}{dx} = 1$.
  - Thus, $\frac{du}{dx} = u = e^x$.
- The exponential function is its own derivative! This means that e^x increases at an extraordinary rate:
  - the bigger it gets, the faster it increases;
  - the faster it increases, the faster its *rate* of increase increases;
  - the faster its rate of increase increases, the faster the rate of increase *of* its rate of increase increases, ...!
- Its growth *compounds* upon itself.

General Real Exponents and Logarithms

- For every positive real number a and every *rational* number r, $a^r = \left(e^{\ln a}\right)^r = e^{r \ln a}$.
- Thus it is natural to define $a^x = e^{x \ln a}$ for any positive real number a and any *real* number x (whether rational or irrational).
- The function $\exp_a : \mathbb{R} \to (0, \infty)$ defined by $\exp_a(x) = a^x$ is differentiable and bijective, except in the special case $a = 1$. (Why?)
- Therefore, $\exp_a$ has a differentiable inverse $\log_a$. It is easy to compute that $\log_a = \frac{\ln}{\ln a}$. (That is, $\log_a(x) = \frac{\ln x}{\ln a}$).

When a Rabbit Meets Another Rabbit and They Fall in Love

- The exponential function is crucial for modeling quantities whose rates of growth are proportional to their sizes. For example, it has many applications to population biology
- Suppose there is a population of, say, $100,000$ rabbits with an unlimited food supply and no predators.
- We wish to study how the population will grow over time.
- Obviously, the population changes one rabbit at a time.
- But with so many rabbits, both the birth of new rabbits and the death of old rabbits will be very frequent, and the step of adding one rabbit is very small compared to the total population. So we can learn a lot by approximating the population of rabbits as a smooth function of time, $p = F(t)$.
- What differentiable function F best models the rabbit population?

Rabbits & More Rabbits & More & More Rabbits

- It is reasonable to assume that, given their unlimited food supply and the absence of predators, the rabbits have a constant birth rate and a constant death rate. Subtracting the death rate from the birth rate gives a constant rate of increase. If we measure time in months, this rate will be in rabbits per rabbit per month.
- Thus, the rate of change in the rabbit population (in rabbits per month) at any instant is proportional to the number of rabbits at that instant.
- For example, let us suppose the birth rate is 2 rabbits per rabbit per month and the death rate is .1 rabbit per rabbit per month. This gives a rate of increase of 1.9 rabbits per rabbit per month.
- So when there are p rabbits, the rate of increase will be $1.9p$ rabbits per month.

& More & More & More Rabbits

- Translating into a differential equation, we obtain

$$\frac{dp}{dt} = 1.9p$$

- Rearranging this equation, we obtain

$$\frac{dp}{p} = 1.9dt$$

- Anti-differentiating, we obtain

$$\ln p = 1.9t + C$$

- At $t = 0$, $p = 100,000$, so $C = \ln(100,000)$; hence $p = e^{1.9t + \ln(100,000)} = e^{\ln(100,000)} e^{1.9t} = 100,000e^{1.9t}$.

$$p = 100,000e^{1.9t}$$

Doubling Time

- How long does it take before the population of rabbits doubles to $200,000$?
- Solving for t in the equation $200,000 = 100,000e^{1.9t}$ yields $t = \frac{\ln 2}{1.9}$.
- Let's use Riemann sums to calculate $\ln 2$. (We will do this in class.)