Local Coordinates & Differentials

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The Correspondence Between Numbers and Linear Functions

- The *number* m completely determines the *function* defined by the linear relationship $dy = mdx$.
- Conversely, the linear function $dy = mdx$ completely determines the number $m = \frac{dy}{dx}$. This number is called the *slope* of the linear function.

$$dy = mdx \Leftrightarrow m = \frac{dy}{dx}$$

- Every real number corresponds to a linear function. Different numbers determine different functions. Every linear function corresponds to a number, its slope.
- That is, there is a *one-to-one correspondence* between real numbers and linear functions.

$$m \leftrightarrow \text{ function defined by } dy = mdx$$

Preview: Linear Functions in Higher Dimensions

- In Calculus 3 and in Linear Algebra, we study functions with higher dimensional inputs and outputs.
- You are already familiar with this concept. For example, the area of a rectangle is a function of its length and width: $A = f(l, w) = lw$. Its input is two-dimensional.
- In general, a linear function is a function for which every term in the formula is linear. For example,

$$f(x, y, z) = (2x + 3y - z, x + y + 5z)$$

- In other words, each output is a linear combination of the inputs.
- Is the area of a rectangle a linear function of its length and width?

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- Is the area of a rectangle a linear function of its length and width? No!

Linear Functions & Matrices

- Just as there is a one-to-one correspondence between one-dimensional linear functions and numbers, there is a one-to-one correspondence between higher dimensional linear functions and *matrices* of real numbers. For example:

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 3y - z \\ x + y + 5z \end{bmatrix}$$

- You can think of a number as a 1×1 matrix!

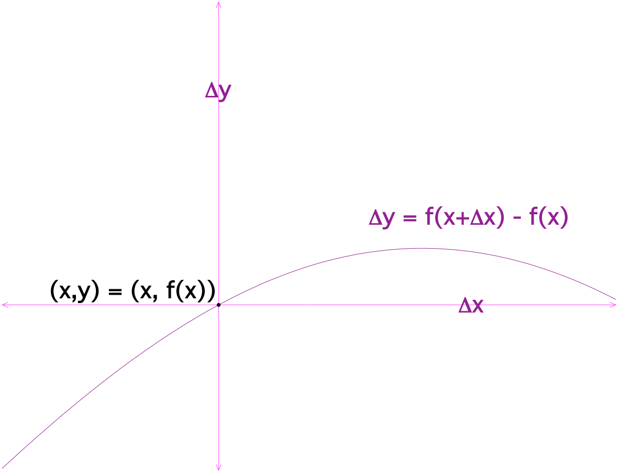
Local Linearity

- Linear functions are relatively simple to understand and have many useful properties.
- As you know from Calculus 1, we *study* more complicated functions *using* linear functions.
- We do this by studying the linear relationship between changes in input and output *at each point* where such a relationship applies.
- If a function is differentiable, its derivative gives this relationship.

Local Coordinates: The $(\Delta x, \Delta y)$-Coordinate System

- Since the linear relationship between inputs and outputs varies from point to point, it is helpful to view this relationship in *local coordinates* based at each point $(x, y) = (x, f(x))$.
- *Two* systems of local coordinates, both with the same scale as the (x, y)-coordinate system, are useful.
- The first is the $(\Delta x, \Delta y)$-coordinate system, in which we graph $\Delta y = f(x + \Delta x) - f(x)$.
- Thus, Δx denotes change in input away from the value x and Δy denotes the corresponding change in the output of f away from the value $y = f(x)$.
- In other words, $y + \Delta y = f(x + \Delta x)$.
- The graph $\Delta y = f(x + \Delta x) - f(x)$ is just a *translation* of the graph of f taking a fixed value (x, y) to the origin.

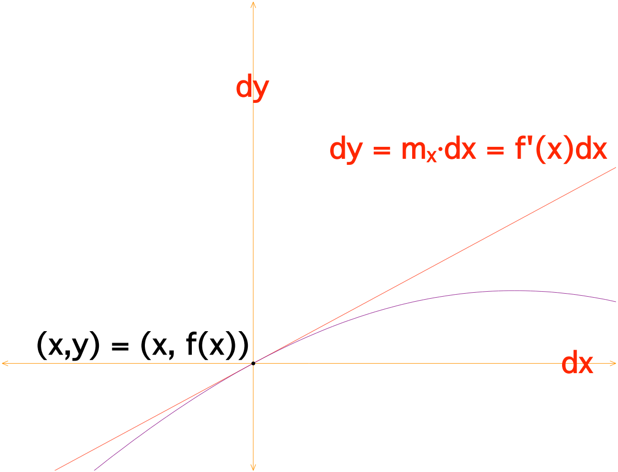
The Graph $\Delta y = f(x + \Delta x) - f(x)$



Local *Differential* Coordinates: The (dx, dy)-Coordinate System

- The second important local coordinate system is the (dx, dy)-coordinate system, in which we graph the linear function $dy = m_x \cdot dx$, where $m_x = f'(x)$ is the slope of the curve $y = f(x)$ at $(x, f(x))$.
- Thus, dx denotes a change in input away from the value x and dy denotes the corresponding *linear* change in accordance with the rate of change at that point.
- The linear change in output dy from y is called the *differential* change in f resulting from a *differential* change in input dx from x.
- For a given change in input, dy and Δy are generally different, unless f is a linear function to begin with. Δy is the *actual* change in the output of f, whereas dy only approximates it near $(x, y) = (x, f(x))$.

The Graph $dy = m_x \cdot dx = f'(x)dx$



Integration of the Differential Changes Gives the Total Change

- To *integrate* means to put together.
- Integration combines the differential changes to obtain the total change.
- Thus, if $y = F(x)$,

$$\int_a^b dy = F(b) - F(a).$$

- That is the content of the Fundamental Theorem of Calculus, interpreted in the language of differentials.
- To compute integrals exactly, we use substitution or some other method to rewrite the differential form being integrated as the differential $dy = F'(x)dx$ of some function F.

Example using Substitution

$$\int_{-\frac{1}{2}}^{0} \sqrt{2x+1}dx = \frac{1}{2}\int_0^1 u^{\frac{1}{2}} du = \frac{1}{2}\int_0^1 d\left(\frac{2}{3}u^{\frac{3}{2}}\right) = \frac{1}{3}u^{\frac{3}{2}}\Big|_0^1 = \frac{1}{3},$$

where $u = 2x+1$; hence $x = \frac{u-1}{2}$ and $dx = \frac{1}{2}du$.

Note how we change the limits of integration to correspond with the new variable.