THE LENGTH OF AN ARC

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Let us recall how we showed that there was a well-defined oriented area under a continuous curve between two points. (By area "under" a curve we mean a number having the properties of oriented area associated to the region between the curve and the independent axis. By "well-defined" we mean that there is just one number satisfying this requirement.) First we estimated the area more and more closely from above and below. Then we showed that, if the curve is continuous, these decreasing over-estimates and increasing underestimates converge to the same limit. That limit must be the area under the curve.

We handled volume in a similar manner, and it too depends on the continuity of the curve. In both cases, the function defining the curve need not be differentiable. Continuity is sufficient; however, a discontinuous curve does not necessarily bound a well-defined area. Once we had established the existence of a well-defined area or volume, similar reasoning showed us how to find a differential form whose integral gives us that area or volume.

We will now apply a similar approach to arc length. Although proving all the details of arc length is beyond the scope of this course, I think you will find the following arguments convincing and illuminating. You will find some blanks spaces to fill in yourself!

1. The arc length of a circle.

Let us consider the simple example of a circle. We know the circumference of the circle is defined and equal to π times the diameter; now let us see why this is so. In the process, we will shed light on how to find the arc length of more general curves.

Consider marking the circle with n equally spaced points, and consider the inscribed and circumscribed polygons touching the circle at these points, as in Figure 1 below.

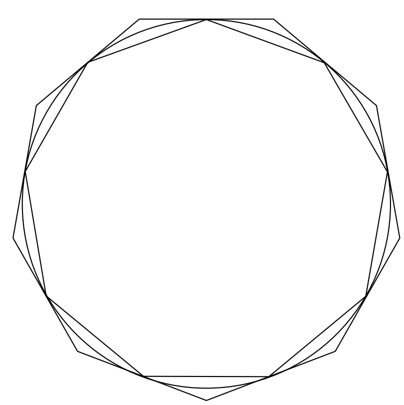

Figure 1. Inscribed and circumscribed polygons of a circle.

If we double the number of points, the perimeter of the inscribed polygon will ________________, since the path along two sides of a triangle (in this case, the congruent sides of an isosceles triangle) is longer than the remaining side (in this case, the base). On the other hand, the perimeter of the circumscribed polygon will ________________, since the apex of each triangle is "cut off." The new inscribed and circumscribed polygons are shown by dotted lines in the Figure 2. Continuing in this manner, we get an increasing sequence of inscribed perimeters and a decreasing sequence of circumscribed perimeters.

It remains to be seen that the increasing inscribed perimeters and the decreasing circumscribed perimeters converge to the same number as n goes to ∞. To see this, observe that if we line up

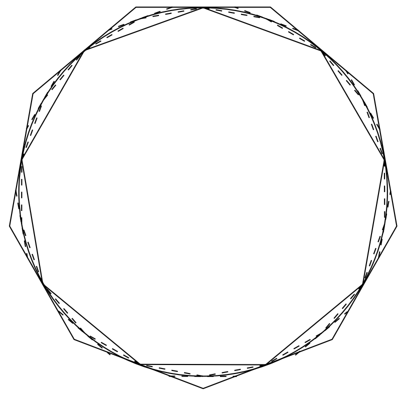

Figure 2. Doubling the number of sides

the triangles bounded by the sides of the inscribed and circumscribed polygons, the segments of the circumscribed polygon may be rotated and translated to lie along the hypotenuse of a right triangle whose base is the perimeter of the inscribed polygon, as shown by Figure 3. As the number of sides increases, the angle θ between the base and hypotenuse decreases to zero, because the slope of the tangents around the circle varies _________________ly. (I other words, as two points on the circle approach each other, the slopes of the tangents at those points also approach the same slope. The angle between the tangent lines becomes arbitrarily close to zero if the points are sufficiently close.) As this happens, the lengths of the base and hypotenuse must approach the same limit.

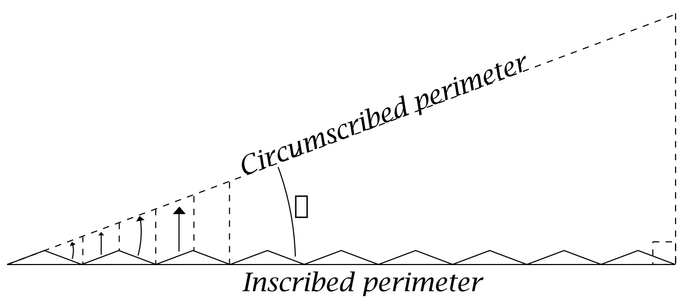

Figure 3. Comparing the inscribed and circumscribed perimeters.

2. The arc length of a more general curve

For a more general curve, we similarly approach from above and below by polygonal paths, as illustrated in Figure 4. Modifications of the argument above show that, if a curve has a _________________ly varying slope – in other words, if the _____________________________ of the function defining the curve is _________________ – then the lengths of these polygonal paths converge to the same limit, giving the curve a well-defined length, called its *arc length*, between any two points on it. If the _________________ of this function does not exist at some point or is dis_________________ at some point, then the part of the curve passing through that point may not have a well-defined length.

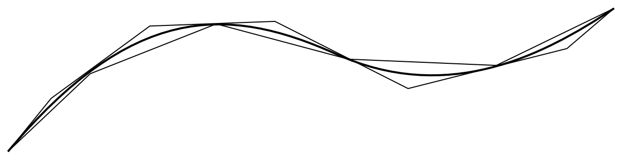

Figure 4. Polygonal approximations to a curve.

3. The differential arc length

The variable s is traditionally used to designate arc length. For simplicity we consider distance along the curve, as opposed to displacement in one direction or the other, so we will view the arc length s as an positive, rather than an oriented, quantity.

Consider a portion of a curve that is increasing (that is, y increases with increasing x) and concave downward, as depicted in Figure 5. Other situations can be handled by a similar analysis, with appropriate modifications. Since, for an increasing curve, each value of x along the curve corresponds to a unique value of y, we may either consider y as a function of x or x as a function of y. Let us provisionally view y as a function of x in order to do a rigorous calculation of the differential arc length, ds; however, the formula we will get in the end for ds can be written in a completely symmetrical fashion with respect to the two variables. We assume, based on the discussion of the previous sections, that this function has a ______________________________ ______________________________. Fill in the blanks as you read the discussion below.

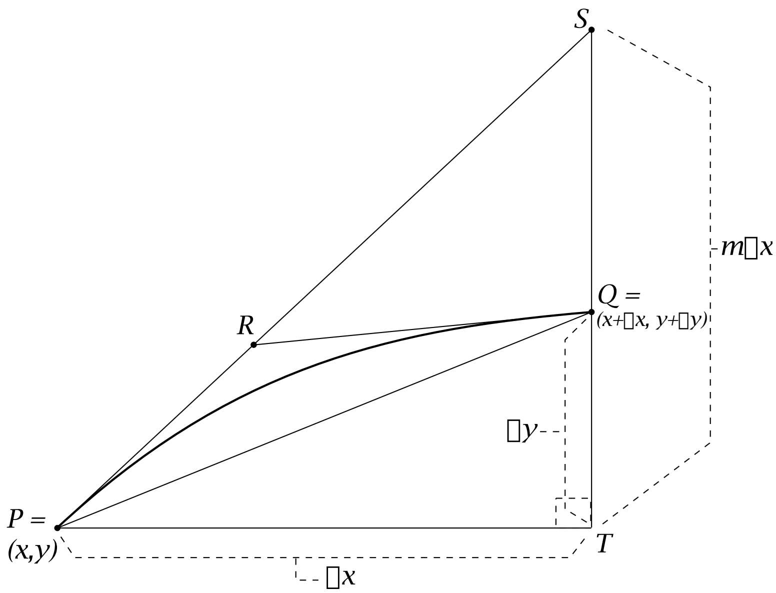

Figure 5. Polygonal approximations to a curve.

Let Δs be the actual length of the curve between P and Q. (Note that without the assumption we just made, Δs might simply not exist. Interestingly, we will not need to use our underlying assumption in any other way in actually calculating ds.) We have $|PQ| < \Delta s < |PR| + |RQ|$, where $|PQ|$ denotes the length of segment PQ, etc. Because y increases with x, $\angle RQS$______$90°$. Since in any triangle the greater side is opposite the greater angle, and a triangle can have at most one obtuse angle, $|RS|$______$|RQ|$. Thus, $|PR| + |RQ|$______$|RS|$, and we have $|PQ| < \Delta s$______$|RS|$.

Since PS is tangent to the curve at $P = (x, y)$, its slope is $\frac{dy}{dx}$; to simplify the notation, let $m = \frac{dy}{dx}$. Thus, $|ST| = m\Delta x$. The slope of segment PQ is $\frac{\Delta y}{\Delta x}$; for simplicity of notation, denote this slope by $\bar{m}$.

By Pythagorean Theorem, we have

$$|PQ| = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 + (\bar{m}\Delta x)^2} = \sqrt{1 + \bar{m}^2}\, \Delta x.$$

Similarly,

$$|PS| = \sqrt{\Delta x^2 + (m\Delta x)^2} = \sqrt{1 + m^2}\, \Delta x.$$

Thus, in terms of Δx and the two slopes $\bar{m}$ and m, we have

______________________________ $< \Delta s <$ ______________________________, from which we obtain

$$\underline{\hspace{5cm}} < \frac{\Delta s}{\Delta x} < \underline{\hspace{5cm}}.$$

But by definition of the derivative

$$\lim_{\Delta x \to 0} \bar{m} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = m.$$

Therefore, by the Squeeze Theorem we obtain that

$$\frac{ds}{dx} = \underline{\hspace{3cm}} \text{ and,hence, } ds = \underline{\hspace{3cm}}.$$

Finally, putting dx back under the radical as dx^2 and noting that $mdx = dy$, we obtain the symmetrical formula

$$ds = \sqrt{dx^2 + dy^2}.$$

I particularly like the formula obtained by squaring both sides (which also allow us to view ds as an oriented quantity, having either a positive or negative value depending on the direction chosen for traversing the curve, if we wish):

$$ds^2 = dx^2 + dy^2.$$

It is easy to visualize this relationship and remember the formula by labeling Figure 6 with the appropriate three differentials.

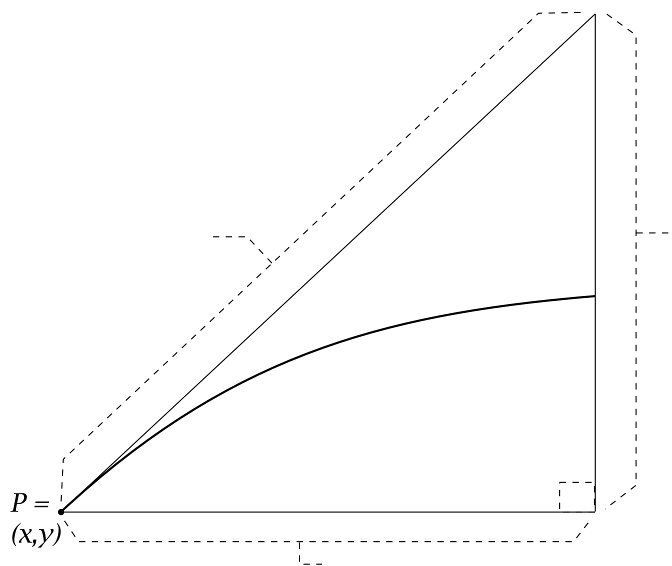

FIGURE 6. The relationship between the differentials dx, dy, and ds.