

Review of
Algebra &
Logic

Charles
Delman

The Language
and Logic of
Mathematics

Propositions
Logic

The Real
Number
System

Relations and
Functions

Exponential
Notation

Review of Algebra & Logic

Charles Delman

August 27, 2015

- 1 The Language and Logic of Mathematics
 - Propositions
 - Logic
- 2 The Real Number System
- 3 Relations and Functions
- 4 Exponential Notation

You Must Make Logical Connections

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Understanding higher mathematics requires making logical connections between ideas.

Please take heed now!

- You cannot learn calculus by memorizing procedures.
You cannot learn calculus by just getting the general idea.
 - The ideas are too big.
 - There are too many new concepts.
 - The problems are too hard.
- You must refine your way of thinking to make it more precise.

An Example from Algebra: the “Old” (Sloppy) Way

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$$x = \sqrt{x + 2}$$

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$$x = \sqrt{x + 2}$$
$$x^2 = x + 2$$

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$$x = \sqrt{x + 2}$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

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$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

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$$x = -1 \text{ or } x = 2$$

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$$x = \sqrt{x+2}$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x+1)(x-2) = 0$$

$$x = -1 \text{ or } x = 2$$

But wait!

$$-1 \neq \sqrt{-1+2} = \sqrt{1} = 1 \quad \textcircled{\times} \quad 2 = \sqrt{2+2} = \sqrt{4} = 2 \quad \checkmark$$

So $x = 2$ is the only solution.

Clear Exposition and Precise Reasoning is the Goal

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How can we explain and justify our solution to the equation?

- How do we describe each step to show what we really did?
- What is the logical relationship among the steps?
- Why did we get the *false* solution $x = -1$?
- How do we know we have *all* the solutions?

A clear, precise, and fully justified solution requires that we use logic!

An Equation is an Open Proposition

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- An equation is a *proposition*.
- A *proposition* is a statement that must be true or false.
 - An opinion is *not* a proposition.
(Example: “Mozart’s music is boring.” Some would say “true”, others would say “false”.)
 - A question is *not* a proposition. It is not a statement.
- An equation states that two numbers are equal. Either they are or they are not. The equation must be either true or false (but not both). (Example: $2 = 5$. This is false.)
- An equation involving variables is an *open* proposition.
 - The value of the variable is left open.
 - Whether the statement is true or false depends on the value of the variable.

Propositions Define Sets

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- Solving an equation means finding *all* values of the variable that make the equation true.
- The set of all such values is called the *solution set* of the equation. Thus, the solution set of the equation $x = \sqrt{x+2}$ is the set $\{2\}$. More formally:

$$\{x : x = \sqrt{x+2}\} = \{x : x = 2\} = \{2\}.$$

These symbols stand for the (true) statement, “The set of numbers x *such that* $x = \sqrt{x+2}$ is equal to the set of numbers x *such that* $x = 2$, which is (of course) the set containing exactly the element 2.” (Two sets are equal if their elements are the same.)

Solving an Equation Requires Logic

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- The terms we just discussed may be familiar.
- The explicit reasoning used to solve the equation may not.
- We must first translate our equation of sets,

$$\{x : x = \sqrt{x+2}\} = \{x : x = 2\},$$

into a pair of *conditional* relationships:

- 1** If $x = \sqrt{x+2}$, then $x = 2$.
This means that 2 is the *only possible* solution.
- 2** If $x = 2$, then $x = \sqrt{x+2}$.
This means that 2 really *is* a solution.

Logical Connectives

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- *Logical connectives* are words or phrases that convey a relationship between the truth or falsity of propositions.
- You need to know six of them:
not, and, or, if (commonly used with *then*), *only if, implies*.
- The meanings of “not” and “and” should be clear from common sense. Examples: “ $2 + 2 = 4$ ” is true; “ $2 + 2 \neq 4$ ” is false; “ $2 + 2 \neq 5$ ” is true; “ $2 + 2 = 4$ and $2 + 3 = 5$ ” is true; “ $2 + 2 = 4$ and $2 + 2 = 5$ ” is false. (The symbol \neq means “does *not* equal”.)
- In mathematical language, “or” is not exclusive. It is used as in “blanket or pillow”, rather than “coffee or tea”: you can have both. Thus, “ $2 + 2 = 4$ or $2 + 3 = 5$ ” is true; “ $2 + 2 = 4$ or $2 + 2 = 5$ ” is also true; “ $2 + 2 = 5$ or $2 + 2 = 6$ ” is false.

Logical Connectives Continued:

Conditional Statements

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- The connectives “if”, “only if”, and “implies” are used to relate open propositions.
- The following (true) propositions are synonymous:
 - If $x = 2$, then $x^2 = 4$.
 - $x = 2$ only if $x^2 = 4$.
 - $x = 2$ implies $x^2 = 4$.
- In the above statements, the *condition* $x = 2$ *forces* – *leads to* – the *consequence* $x^2 = 4$.
- *Important note:* Nothing is conveyed about forcing – that is, implication – in the *other* direction. In fact, the statement “If $x^2 = 4$, the $x = 2$ is false. (x could be -2 .)”

Logical Connectives Continued: Convenient Abbreviations & Symbols

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- The following (true) propositions are synonymous:
 - If $x = 2$, then $x^2 = 4$.
 - $x = 2$ only if $x^2 = 4$.
 - $x = 2$ implies $x^2 = 4$.
 - $x = 2 \Rightarrow x^2 = 4$.
- The following (true) propositions are synonymous:
 - If $x = 2$, then $x + 1 = 3$, and if $x + 1 = 3$, then $x = 2$.
 - $x + 1 = 3$ if $x = 2$, and $x + 1 = 3$ only if $x = 2$.
 - $x + 1 = 3$ if and only if $x = 2$.
 - If $x + 1 = 3$, then $x = 2$, and if $x = 2$, then $x + 1 = 3$.
 - $x = 2$ if $x + 1 = 3$, and $x = 2$ only if $x + 1 = 3$.
 - $x = 2$ if and only if $x + 1 = 3$.

Logical Connectives Continued: One More Convenient Symbol

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- The following (true) propositions are synonymous:
 - $x = 2 \Rightarrow x + 1 = 3$ and $x + 1 = 3 \Rightarrow x = 2$.
 - $x + 1 = 3 \Rightarrow x = 2$ and $x = 2 \Rightarrow x + 1 = 3$.
 - $x = 2 \Leftrightarrow x + 1 = 3$.
 - $x + 1 = 3 \Leftrightarrow x = 2$.
- The propositions $x + 1 = 3$ and $x = 2$ are *equivalent*.

Practice: Determine Which Propositions are True

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1 $2 > 5$ or $2 < 5$.

2 $2 > 5$ and $2 < 5$.

3 $2 < 5$ or $3 < 5$.

4 $2 < 5$ and $3 < 5$.

5 $4 < 5$ and $5 \not< 5$.

6 $x < 5 \Rightarrow x + 1 < 5$.

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True

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True

False

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- | | | |
|-----------|-------------------------------------|-------|
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Equal Sets are Defined by Equivalent Propositions

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- We can now summarize our solution to the equation logically and succinctly in symbols:

$$x = \sqrt{x + 2} \Leftrightarrow x = 2$$

- The propositions $x = \sqrt{x + 2}$ and $x = 2$ are *equivalent*.
- To fully understand *how* we know they are equivalent, we must understand operations in the *real number system*.

What is a Real Number?

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- The somewhat misleading name *real* number has nothing to do with the existence of these numbers; all numbers exist as concepts, including the (equally misnamed) imaginary numbers and the complex numbers.
- Historically, the name “real” probably arose because these numbers describe physical quantities such as length, area, volume, and mass. Since complex numbers also play a vital role in describing physical phenomena such as waves, this justification really doesn't hold up.
- The real number system is the smallest system big enough to use for calculus.
- We say *system* because it is not just a set of numbers; there are also operations on the numbers.

Number operations

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- Two basic binary *operations*: addition and multiplication.
- An *operation* associates an output to a set of inputs in such a way that the inputs *determine* the output. That is, there is *one and only one output* that goes with the inputs.
- As a contrasting example to help explain this, the association of a daughter to a couple is *not* an operation:
 - The couple may not have a daughter; hence, there is no output.
 - The couple may have more than one daughter; hence, the inputs (parents) do not determine a unique output.
- An operation is really just a special name for a basic type of *function*.

Properties of Operations with Real Numbers

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- Addition and multiplication are *binary* operations: they associate an output to *two* inputs.
- Addition and multiplication are *commutative* operations: the order of the inputs does not affect the output. That is:
 - If x, y , and z are (any) real numbers, $x + y = y + x$.
 - If x, y , and z are (any) real numbers, $xy = yx$.
- Addition and multiplication are *associative* operations: the grouping of three or more inputs into pairs does not affect the ultimate output. That is:
 - If x, y , and z are real numbers, $(x + y) + z = x + (y + z)$.
For example, $(1 + 2) + 3 = 3 + 3 = 6 = 1 + 5 = 1 + (2 + 3)$
 - If x, y , and z are real numbers, $(xy)z = x(yz)$.
- Since grouping doesn't matter, we can leave it out, simply writing $x + y + z$ and xyz .

Operations Illustrated

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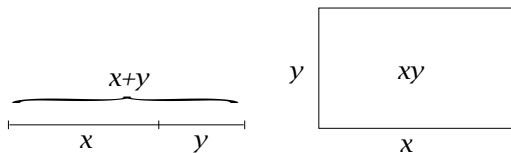
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Properties of Operations on Real Numbers Continued

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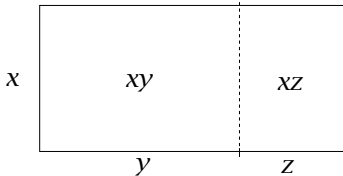
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- Please note that addition and multiplication are *not* associative in combination. For example, $(1 + 2) \cdot 3 \neq 1 + (2 \cdot 3)$.
- However, these operations do have a property in combination: the *distributive* property. Multiplication *distributes* over addition:

If x , y , and z are real numbers, $x(y + z) = xy + xz$.



$$x(y+z) = xy + xz$$

Existence of Identities and Inverses: Additive

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- The special number 0 is the *additive identity*:
Adding the quantity 0 does not change the total quantity, thus adding 0 to any number gives the same number.
That is, if x is any real number, then $x + 0 = x$.
- Every real number has a unique *additive inverse*:
When additive inverses are added, the result is 0.
 - Example: $5 + (-5) = 0$.
 - 0 is its own additive inverse: $0 + 0 = 0$.
 - $-(x + y) = (-x) + (-y)$, since
$$x + y + (-x) + (-y) = x + (-x) + y + (-y) = 0 + 0 = 0$$
- *Inversion* is a *unitary* operation: it requires only one input.

An Important Consequence

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- If x is any real number, then $0x = 0$. Here is a proof based on the previous properties:
 - $0x = (0 + 0)x$, since $0 + 0 = 0$.
 - $(0 + 0)x = 0x + 0x$, by the distributive property.
 - Thus, $0x = 0x + 0x$.
- Whatever $0x$ is, it has an additive inverse, $-(0x)$. When we add pairs of equal numbers, we get the same output; hence
 - $0 = 0x + (-(0x)) = 0x + 0x + (-(0x)) = 0x + 0 = 0x$.
 - Thus $0x = 0$

Tip: If that was hard to follow, write it out for yourself! Often, when reading math, you need to do that.

Existence of Identities and Inverses: Multiplicative

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- The special number 1 is the *multiplicative identity*:
Multiplying any number by 1 gives the same number. In physical terms, for example, if a room is 3 meters long and 1 meter wide, it will have a floor area of 3 square meters.
- Every real number *except* 0 has a unique *multiplicative inverse*:

When multiplicative inverses are multiplied, the result is 1.

- Example: $(5) \left(\frac{1}{5}\right) = 1$.
- 1 is its own multiplicative inverse: $(1)(1) = 1$.
- $\frac{1}{xy} = \left(\frac{1}{x}\right) \left(\frac{1}{y}\right)$, since
$$(xy) \left(\frac{1}{x}\right) \left(\frac{1}{y}\right) = x \left(\frac{1}{x}\right) y \left(\frac{1}{y}\right) = (1)(1) = 1.$$
- Zero cannot have multiplicative inverse because, as we have seen, if x is any real number, $0x = 0$. There is *no* real number x for which $0x = 1$; that is impossible.

An Important Consequence

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- If $xy = 0$, then $x = 0$ or $y = 0$:
 - We will show, under the given hypothesis, that if x is not zero, then y is.
 - This shows that one or the other must be zero.
 - If $x \neq 0$, then it has a multiplicative inverse, $\left(\frac{1}{x}\right)$. Thus:

$$y = 1y = \left(\frac{1}{x}\right)xy = \left(\frac{1}{x}\right)0 = 0.$$

- We use this fact often when solving quadratic equations!

Defined Operations

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- *Subtraction* is defined to be addition of the additive inverse:
If x and y are real numbers, $x - y = x + (-y)$.
- *Division* is defined to be multiplication by the multiplicative inverse:
If x and y are real numbers, and $y \neq 0$, $\frac{x}{y} = x \left(\frac{1}{y}\right)$.
- Since 0 has no multiplicative inverse, division by 0 is not defined.
- Many other operations and functions may be defined using the basic ones. For example, x^2 is defined by $x^2 = (x)(x)$. Squaring is a *unitary* operation, since it only takes one input, which is used for both factors.

Using the Properties of Operations

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- The properties of addition and multiplication are clear from examples and pictures.
- Why do we specify them in general abstract terms?
- Undoubtedly, you did not need to know the commutative, associative, or distributed property to understand, as a small child, that $2 + 3 = 5$ or that $2 + 3 + 5 = 10$.
- We specify these properties in order to correctly work with expressions involving variables (unknowns).
- It is easy to calculate with specific numbers, but harder to calculate with unknown numbers.
- Variables represent numbers. The properties we have specified tell us how numbers behave, even if we don't know which specific numbers the variables might represent.

Practice: Simplify Each Expression

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$$1 \quad \frac{(2)(3) + 3}{3}$$

$$2 \quad \frac{xy + y}{y}$$

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$$1 \quad \frac{(2)(3) + 3}{3} = \frac{(6 + 3)}{3} = \frac{9}{3} = 3$$

$$2 \quad \frac{xy + y}{y}$$

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- 1** Using instead the multiplicative identity, distributive property, definition of division, and associativity provides a more illuminating method:

$$\begin{aligned}\frac{(2)(3) + 3}{3} &= \frac{(2)(3) + (1)(3)}{3} = \frac{(2 + 1)(3)}{3} \\ &= \frac{(3)(3)}{3} = (3)(3) \left(\frac{1}{3}\right) = (3)(1) = 3\end{aligned}$$

2 $\frac{xy + y}{y}$

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1

$$\frac{(2)(3) + 3}{3} = \frac{(2 + 1)(3)}{3} = (3)(3) \left(\frac{1}{3}\right) = 3$$

- 2** While more longwinded than needed for working with specific numbers, this deeper understanding helps us correctly simplify the second expression:

$$\frac{xy + y}{y} = \frac{(x + 1)y}{y} = (x + 1)(y) \left(\frac{1}{y}\right) = x + 1$$

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$$3 \quad \frac{2}{3} + \frac{3}{5}$$

$$4 \quad \frac{x}{y} + \frac{y}{z}$$

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3

$$\begin{aligned}\frac{2}{3} + \frac{3}{5} &= \left(\frac{2}{3}\right) \left(\frac{5}{5}\right) + \left(\frac{3}{5}\right) \left(\frac{3}{3}\right) = \\ (2)(5) \left(\frac{1}{3}\right) \left(\frac{1}{5}\right) + (3)(3) \left(\frac{1}{5}\right) \left(\frac{1}{3}\right) \\ &= (10 + 9) \left(\frac{1}{15}\right) = \frac{19}{15}\end{aligned}$$

4

$$\frac{x}{y} + \frac{y}{z}$$

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$$3 \quad \frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{19}{15}$$

$$4 \quad \frac{x}{y} + \frac{y}{z} = \frac{xz}{yz} + \frac{y^2}{yz} = \frac{xz + y^2}{yz}$$

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5 $\sqrt{4 + 9}$

6 $\sqrt{x^2 + y^2}$

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$$5 \quad \sqrt{4 + 9} = \sqrt{13}$$

Note that $\sqrt{4} + \sqrt{9} = 2 + 3 = 5 \neq \sqrt{13}$. ($5^2 = 25$.)

$$6 \quad \sqrt{x^2 + y^2} \text{ cannot be simplified.}$$

Our Example from Algebra: Logical Relationships

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- First, let's fill in the logical relationships among the equations.

$$x = \sqrt{x + 2} \Rightarrow \quad (1)$$

$$x^2 = x + 2 \Leftrightarrow \quad (2)$$

$$x^2 - x - 2 = 0 \Leftrightarrow \quad (3)$$

$$(x + 1)(x - 2) = 0 \Leftrightarrow \quad (4)$$

$$x = -1 \text{ or } x = 2 \quad (5)$$

- The relationship between equation (1) and equation (2) is the weak link; since the implication goes only one way, our argument shows only that $x = \sqrt{x + 2} \Rightarrow x = -1 \text{ or } x = 2$.

Our Example from Algebra: Justifying our Logic

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- Next, we will justify each logical relationship.
- The implication

$$x = \sqrt{x + 2} \Rightarrow \quad (1)$$

$$x^2 = x + 2 \quad (2)$$

is true because the expressions on each side of equation (1) represent the same number; therefore, we get the same result on each side when we square that number.

- We cannot assume the reverse implication is true. Recall that $\sqrt{\quad}$ means the *positive* square root. If x is negative, $\sqrt{x^2} = -x$. The converse implication is true for positive values of x , but not for negative values.

Our Example from Algebra: Justifying our Logic

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- The biconditional (that is, two-way) implication

$$x^2 = x + 2 \Leftrightarrow \quad (2)$$

$$x^2 - x - 2 = 0 \quad (3)$$

is true because if we add $-x - 2$ to the equal sides of equation (2) we get the same result and, conversely, if we add $x + 2$ to the equal sides of equation (3), we get the same result.

Our Example from Algebra: Justifying our Logic

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- The biconditional (that is, two-way) implication

$$x^2 - x - 2 = 0 \Leftrightarrow \quad (3)$$

$$(x + 1)(x - 2) = 0 \quad (4)$$

is true because we have simply used the distributive property to rewrite the expression on the left side of equation (3) in a different way.

- We did so in order to use our result, shown earlier, that if the product of two real numbers is zero, then at least one of the numbers must be zero. (The converse, that if one of the two factors is zero, then the product is zero, is also obviously true.)

Our Example from Algebra: Justifying our Logic

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- Using the fact that the product of two numbers is zero if and only if one of the two factors is zero, we obtain

$$(x + 1)(x - 2) = 0 \Leftrightarrow \quad (4)$$

$$x = -1 \text{ or } x = 2 \quad (5)$$

- We now know that -1 and 2 are the only possible solutions, and we have explained why. Since our earlier reasoning showed that the converse of the implication relating equations (1) and (2) is false for negative values of x and true for positive values, we know that -1 is not a solution, but 2 is a solution.
- Thus, we have justified that $x = \sqrt{x + 2} \Leftrightarrow x = 2$.

Example of a Relation

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- Suppose a man has 12 dollars to spend on dinner, dessert, and tip at the local diner.
- Being fair-minded, he plans to leave a 2 dollar tip, so he has 10 dollars to spend on food.
- The entrees are chicken, for 6 dollars, catfish, for 7 dollars, and strip steak, for 9 dollars.
- The desserts are pudding, for 1 dollar, pie, for 2 dollars, and chocolate cake, for 4 dollars.
 - What are his possible (dinner, dessert) price combinations, assuming that he has one of each?
 - Graph these combinations in (x, y) coordinates, letting x be the price paid for dinner and y the price paid for dessert.

What Exactly is a Relation?

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- The constraint on our hero's finances clearly engenders a relationship between the amount spent on dinner and the amount spent on dessert: $x + y \leq 12$.
- It is also important to know the possible values of x (6, 7, or 9) and y (1, 2, or 4) that may be considered.
- Mathematicians use the word *relation* to describe this relationship together with the possible values of the variables. (This jargon lets you know they are doing mathematics and not writing a romantic advice column.)
- The set of possible values of the first variable is called the *domain* of the relation, and the set of possible values of the second variable is called the *range*.

Another Relation

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- Note that the relationship in the previous example is not sufficiently strict that you can be sure of what he had for dessert if you know what he had for an entree. For example, if he had chicken, he could have had any of the desserts.
- Here is another relation in which the value of the first coordinate does not determine the value of the second:
$$x^2 + y^2 = 1.$$
- Graph the relation $x^2 + y^2 = 1$ in the (x, y) -coordinate plane. (Hint: What is the distance from a point (x, y) to the origin $(0, 0)$.)

The Pythagorean Theorem

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- The Pythagorean Theorem, which has been known since ancient times, gives an important relationship between length and area.
- The Pythagorean Theorem states that:
 - For a *right triangle in a flat plane* with legs of length a and b and hypotenuse of length c ,
 - $a^2 + b^2 = c^2$.

Why the Pythagorean Theorem is true

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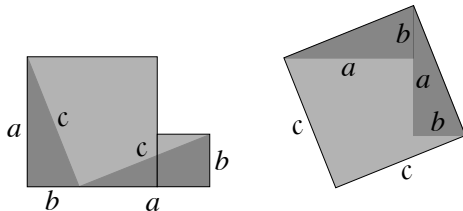
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$$a^2 + b^2 = c^2$$



The Converse of the Pythagorean Theorem

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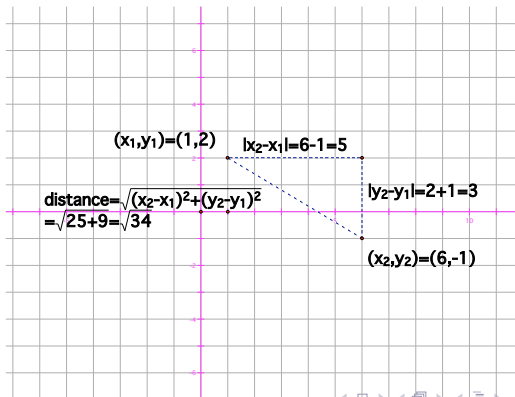
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- Since triangles are rigid - the angles of a triangle are determined by the sides - the *converse* of the above statement must also be true:
 - In a triangle in a flat plane with sides of length a , b , and c , if $a^2 + b^2 = c^2$, then the angle opposite the side of length c (the longest side) is a right angle.

The Pythagorean Theorem and Distance

- The Pythagorean Theorem allows us to calculate distances in the coordinate plane in any direction.
- The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



The Equation of a Circle

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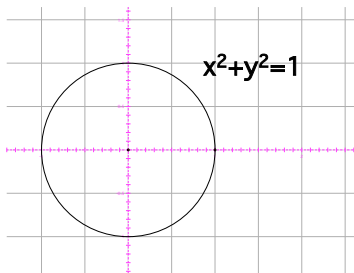
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- Thus, an equation of the form $(x - a)^2 + (y - b)^2 = r^2$ says, “The distance from (x, y) to (a, b) is r .” The letters a , b , and r represent fixed quantities (constants).
- Therefore, its graph is the circle with center (a, b) and radius r .
- In particular, $x^2 + y^2 = 1$ is the *unit circle*: the circle of radius 1 centered at the origin $(0, 0)$.



A Relation that is a Function

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- Suppose now the following additional fact: our hero has a drinking problem and is trying to quit. He is determined to spend at least 11 dollars on dinner, because then his remaining funds will not be sufficient to buy a drink. (He could leave a larger tip, but he doesn't want to raise the waiter's expectations!)
 - What are his possible (dinner, dessert) price combinations (again assuming that he has one of each)?
 - Graph these combinations in (x, y) coordinates, letting x be the price paid for dinner and y the price paid for dessert.

What Exactly is a Function?

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- Now the relation is sufficiently strict that his dessert is determined by what he has for an entree.
 - If he has the chicken, he must get the chocolate cake, or he will not have spent enough.
 - If he has the catfish, he must get the pie. The cake is too expensive, but the pudding is too cheap.
 - And if he has the steak, he can only afford the pudding.
- Mathematicians say that the price of his dessert is a *function* of the price of his entree.
- Notice that it was important to know the possible values of x and y in order to determine the relation precisely and recognize that it is a function. A common mistake among students is to focus only on formulas and ignore the domain and range of a relation.

Functional Notation

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- You may recall *functional notation*, in which a symbol such as f is used to represent the functional relationship, and we write $y = f(x)$.
- It is important to understand that f is not a number, and $f(x)$ is not a function! f is a relationship between numbers.
- x is a number (the input).
- y and $f(x)$ represent the same number (the unique output that corresponds to x).

Calculus is a Method of *Analyzing* Functions

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- Many relations are described by algebraic formulas such as $x^2 + y^2 = 1$ or $f(x) = \sqrt{x^2 - x + 1}$.
- These are called *algebraic* relations.
- Other relations cannot be described by algebraic formulas.
- These are called *transcendental* relations.
- For example, $\sin \theta$ and $\cos \theta$ are not algebraic functions of θ : there is no algebraic formula that expresses the geometric relationships defined in trigonometry.
- Calculus is a branch of the *analysis* of functions, both algebraic and transcendental. Relations that are not functions also play a role in problems whose solutions involve calculus, but functions are central.

Practice Simplifying Expressions Involving Functional Notation

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- 1 Let $f(x) = x^2$. Compute $f(-2)$.
- 2 Let $f(x) = x^2$. Let $u = 3x + 2$. Compute $f(u)$ for $x = -1$
- 3 Let $f(x) = x^2$. Let $u = 3x + 2$. Compute $f(u)$ in terms of x .
- 4 Let $f(x) = x^2$. Compute $f(3x + 2)$.
- 5 Let $f(x) = x^2$. Let Δx be another independent variable. (Δx is a two-letter word representing a single number.)
Let $u = x + \Delta x$. Compute $f(u)$ for $x = -2$ and $\Delta x = 1$.
- 6 Let $f(x) = x^2$. Let Δx be another independent variable.
Let $u = x + \Delta x$. Compute $f(u)$ in terms of x and Δx .
- 7 Let $f(x) = x^2$. Let Δx be another independent variable.
Compute $f(x + \Delta x)$.

Solutions

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- 1** $f(-2) = (-2)^2 = 4.$
- 2** If $x = -1$, then $u = 3(-1) + 2 = -1$. (Note that u is a function of x !) Hence, $f(u) = u^2 = (-1)^2 = 1$ if $x = -1$. Alternatively, $f(u) = f(-1) = (-1)^2 = 1.$
- 3** In general, $f(u) = u^2 = (3x + 2)^2 = 9x^2 + 12x + 4.$ Alternatively, $f(u) = f(3x + 2) = (3x + 2)^2 = \dots.$
- 4** $f(3x + 2) = (3x + 2)^2 = 9x^2 + 12x + 4.$
- 5** If $x = -2$ and $\Delta x = 1$, then $u = -2 + 1 = -1$, hence $f(u) = u^2 = (-1)^2 = 1.$ (Or $f(u) = f(-1) = (-1)^2 = 1.$)
- 6** In general, $f(u) = u^2 = (x + \Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2.$ Alternatively, $f(u) = f(x + \Delta x) = (x + \Delta x)^2 = \dots.$
- 7** $f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2.$

More Practice Simplifying Expressions Involving Functional Notation

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- 1 Let $f(x) = \frac{1}{x}$. Compute $f(-2)$.
- 2 Let $f(x) = \frac{1}{x}$. Let $u = 3x + 2$. Compute $f(u)$ for $x = -1$.
- 3 Let $f(x) = \frac{1}{x}$. Let $u = 3x + 2$. Compute $f(u)$ in terms of x .
- 4 Let $f(x) = \frac{1}{x}$. Compute $f(3x + 2)$.
- 5 Let $f(x) = \frac{1}{x}$. Let Δx be another independent variable. (Δx is a two-letter word representing a single number.) Let $u = x + \Delta x$. Compute $f(u)$ for $x = -2$ and $\Delta x = 1$.
- 6 Let $f(x) = \frac{1}{x}$. Let Δx be another independent variable. Let $u = x + \Delta x$. Compute $f(u)$ in terms of x and Δx .
- 7 Let $f(x) = \frac{1}{x}$. Let Δx be another independent variable. Compute $f(x + \Delta x)$.

Practice Analyzing Polynomial Functions: Sketch the Graph of Each Function

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- $f(x) = x - 2$
- $f(x) = 3(x - 2)$
- $f(x) = 3(x - 2) + 1$
- $f(x) = (x - 1)(x - 2)$
- $f(x) = (x - 1)^2$
- $f(x) = (x - 1)^2 + 1$
- $f(x) = (x - 1)(x - 2)(x - 3)$
- $f(x) = (x - 1)(x - 2)^2$
- $f(x) = x(x - 1)(x - 2)(x - 3)$
- $f(x) = x(x - 1)(x - 2)^2$
- $f(x) = x(x - 2)^3$

Practice Analyzing Polynomial Functions: Find Formulas for the functions f and g

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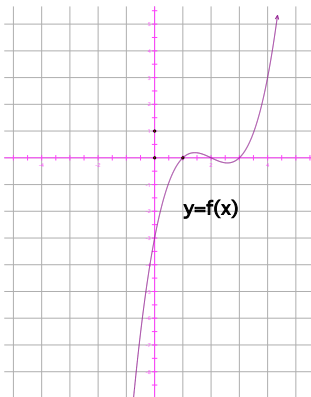
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Practice Analyzing Rational Functions

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- If the domain and range of a function are not stated, it is implicit that the domain consists of all possible inputs to which the formula applies and that the range consists of all possible outputs that result.
- Find the domain and range of the function defined by $f(x) = \frac{x-3}{x+7}$.
- Sketch the graph of f .

Integer Exponents

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- Exponential expressions do not represent a new type of number; they are just a convenient notation.
- Positive integer exponents count the number of repeated factors to provide an abbreviation: $(2)(2)(2) = 2^3$, $(x)(x)(x)(x) = x^4$. The repeated factor is the *base*.
- Clearly, multiplying by the base adds 1 to the exponent, whereas dividing by the base subtracts 1 from the exponent: $x^4x = x^5$, $\frac{x^4}{x} = x^3$.
- If we continue to divide, we get a natural definition for expressions with non-positive integer exponents:
$$x^0 = x^{1-1} = \frac{x}{x} = 1, \quad x^{-1} = x^{0-1} = \frac{x^0}{x} = \frac{1}{x}, \dots$$

The Fundamental Properties of Exponents

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- Another way to look at these definitions is that exponential notation is designed to take advantage of two fundamental properties that facilitate simplifying expressions:
 - $x^m x^n = x^{m+n}$. Example:
 $x^2 x^3 = [(x)(x)][(x)(x)(x)] = x^{2+3} = x^5$
 - If expressions with the same base and exponent are used as repeated factors, we can simply multiply:
 $(x^2)^3 = (x^2)(x^2)(x^2) = x^{2+2+2} = x^{3 \cdot 2} = x^6$.
- The general definition of integer exponents extends these properties:
 - $x^0 x^m = x^{0+m} = x^m$; hence, x^0 must be defined to equal 1, the multiplicative identity.
 - $x^{-m} x^m = x^{-m+m} = x^0 = 1$; hence, x^{-m} must be defined to equal $\frac{1}{x^m}$, the multiplicative inverse of x^m .

Defining Rational Exponents

In Accordance with these Properties

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Notation

- Using these properties, we can further extend the notation to define rational exponents in general:
- $(x^{\frac{1}{2}})^2 = x^{2 \cdot \frac{1}{2}} = x^1 = x$; hence, $x^{\frac{1}{2}}$ must be defined to equal a square root of x , which by convention we take to be the positive square root, \sqrt{x} . (Naturally, $x^{\frac{1}{2}}$ is only defined for positive values of x .)
- Similarly, $(x^{\frac{1}{3}})^3 = x^{3 \cdot \frac{1}{3}} = x^1 = x$; hence, $x^{\frac{1}{3}}$ must be defined to equal the cube root of x , $\sqrt[3]{x}$.
- In general, $(x^{\frac{1}{n}})^n = x^{n \cdot \frac{1}{n}} = x^1 = x$; hence, $x^{\frac{1}{n}}$ is defined to equal the n^{th} root of x , $\sqrt[n]{x}$, positive if n is even. (Thus, $x^{\frac{1}{n}}$ is only defined when this root exists; in particular, $x^{\frac{1}{n}}$ is not defined when n is even and x is negative.)
- More generally, $x^{\frac{m}{n}} = x^{m \cdot \frac{1}{n}} = \sqrt[n]{x^m} = (x^{\frac{1}{n}})^m = (\sqrt[n]{x})^m$.

Simplifying and Factoring Exponential Expressions

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$$\blacksquare \left(\frac{x^{-\frac{1}{3}}}{y^{-\frac{2}{3}}} \right) \left(\frac{x^2}{y^{\frac{1}{3}}} \right)^3 = (x^{-\frac{1}{3}} y^{\frac{2}{3}})(x^2 y^{-\frac{1}{3}})^3 = \\ (x^{-\frac{1}{3}} y^{\frac{2}{3}})(x^6 y^{-1}) = x^{\frac{17}{3}} y^{-\frac{1}{3}}$$

$$\blacksquare x^2(x+1)^{\frac{2}{3}} - 2x(x+1)^{-\frac{1}{3}} = x(x+1)^{-\frac{1}{3}}[x(x+1) - 2] = \\ x(x+1)^{-\frac{1}{3}}(x^2 + x - 2) = x(x+1)^{-\frac{1}{3}}(x+2)(x-1) = \\ \frac{x(x+2)(x-1)}{(x+1)^{\frac{1}{3}}}.$$

- What are all the roots of the function $f(x) = x^2(x+1)^{\frac{2}{3}} - 2x(x+1)^{-\frac{1}{3}}$?

Solution to $x^2(x + 1)^{\frac{2}{3}} - 2x(x + 1)^{-\frac{1}{3}} = 0$

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Using the distributive property, we factor out the lowest power of each factor that appears. The lowest power of x is 1; the lowest power of $x + 1$ is $-\frac{1}{3}$. Thus:

$$x^2(x + 1)^{\frac{2}{3}} - 2x(x + 1)^{-\frac{1}{3}} = x(x + 1)^{-\frac{1}{3}}[x(x + 1) - 2]$$

Next, we distribute, combine like terms, and factor completely:

$$\begin{aligned}x(x + 1)^{-\frac{1}{3}}[x(x + 1) + 2] &= x(x + 1)^{-\frac{1}{3}}[x^2 + x - 2] \\ &= x(x + 1)^{-\frac{1}{3}}(x + 2)(x - 1)\end{aligned}$$

A negative power of a real number cannot be 0, since it is the multiplicative inverse of another number. (In particular, $(x + 1)^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{x+1}} \neq 0$ for all possible values of x .)

Solution to $x^2(x + 1)^{\frac{2}{3}} - 2x(x + 1)^{-\frac{1}{3}} = 0$ (Conclusion)

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Thus,

$$\begin{aligned}x^2(x + 1)^{\frac{2}{3}} - 2x(x + 1)^{-\frac{1}{3}} &= 0 \\ \Leftrightarrow x(x + 1)^{-\frac{1}{3}}(x + 2)(x - 1) & \\ \Leftrightarrow x = 0 \text{ or } x + 2 = 0 \text{ or } x - 1 = 0 & \\ \Leftrightarrow x = 0 \text{ or } x = -2 \text{ or } x = 1 &\end{aligned}$$

Master this technique and keep it in mind, as it is very useful. Note that it is a natural extension to fractional powers of the technique you already know for solving equations by factoring. It relies on the same fundamental fact:

a product is zero if and only if one of the factors is zero.