Written Assignment 1

Due by the beginning *of class on Monday, August 28, 2017.*

The following instructions hold for this and all future assignments:

- Although I urge you to first work the problems in the order that works for you – if you get stuck, go on to another problem and come back later – prepare *neat, well-organized* solutions, revising them as needed, in a **final draft**, with solutions arranged in the order the problems are given, grouped by numbered section. Why? So you can find them when it is time to study! You need not save your scratch work.
- For the same reason, save your assignments in an orderly fashion.
- All solutions should be given in full sentences, arranged into coherent paragraphs as needed, using proper grammar, usage, and syntax for both English and symbolic mathematics. Explain what you did, why you did it, and why it works. (For problems requiring a similar approach, you need only explain it once.) Give the logical connections between steps, and justify every step (except arithmetic).
- Solutions to equations or inequalities should be written symbolically and *include the logical relationships between the steps.*

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1 Logarithmic & Exponential Functions; Integral Definition of the Natural Logarithm

Recall that the natural logarithm function is defined by the formula $\ln x = \displaystyle\int_1^x \frac{dt}{t}$, for $x > 0$. (As with the trigonometric functions, the parentheses around the input variable are traditionally omitted.)

1. What is $\ln 1$?
2. Explain why ln is a strictly increasing function. (Recall that for a function f to be *strictly increasing* means that, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. Hint: Consider the representation of ln as the oriented area under a curve. Alternatively, consider what you know about derivatives and the Fundamental Theorem of Calculus.)

3. Use the definition to show that $\ln xy = \ln x + \ln y$. [Hint: Break the integral $\ln xy = \displaystyle\int_1^{xy} \frac{dt}{t}$ into a sum of two integrals by separating the interval $[1, xy]$ at x. Then do a suitable substitution in one of them.]

4. What is $\lim_{x\to+\infty} \ln x$? (Hint: $\ln 4 = \ln 2 + \ln 2$; $\ln 8 = \ln(2\cdot 2\cdot 2) = \ln 2 + \ln 2 + \ln 2$; ....)

5. Explain why $\ln \frac{1}{2} = -\ln 2$. (Hint: $\left(\frac{1}{2}\right)(2) = 1$.) More generally, explain why for any $x > 0$, $\ln \frac{1}{x} = -\ln x$.

6. What is $\lim_{x\to 0^+} \ln x$? (Hint: $\ln \frac{1}{4} = \ln \frac{1}{2} + \ln \frac{1}{2}$; ....)

7. Use the definition to show that $\ln x^a = a \ln x$. [Hint: Substitute $u = t^{1/a}$; hence, $t = u^a$. What is dt? What are the new limits of integration?] A fine point: we do not yet know how to define x^a if a is irrational; we will see the correct definition soon.

   Make sure you remember how to define rational exponents in general! Do we need to talk about this in class?

8. Explain using areas and the definition of the natural logarithm why $\ln x < x - 1$ for $x > 1$. Since $\ln x \le 0$ for $0 < x \le 1$, it follows that $\ln x < x$ for any positive value of x.

9. What is $\ln'(1)$, where $\ln'$ denotes the derivative of ln? What is $\lim_{x\to+\infty} \ln'(x)$? What is $\lim_{x\to 0^+} \ln'(x)$? Is there any point at which $\ln x \le 0$?

10. Sketch the graph of $y = \ln x$.

11. Explain why $\lim_{h\to 0} \frac{\ln(1+h)}{h} = 1$. [Hint: This limit is the derivative of a particular function at a particular point. What is the function and what is the point?]

12. According to Kathryn Schulz, writing recently in the *New Yorker* magazine:

    "For decades, seismologists had believed that Japan could not experience an earthquake stronger than magnitude 8.4. In 2005, however, at a conference in Hokudan, a Japanese geologist named Yasutaka Ikeda had argued that the nation should expect a magnitude 9.0 in the near future – with catastrophic consequences, because Japans famous earthquake-and-tsunami preparedness, including the height of its sea walls, was based on incorrect science."

    "The Really Big One," by Kathryn Schulz, *The New Yorker*, July 20, 2015.

    (http://www.newyorker.com/magazine/2015/07/20/the-really-big-one)

    Unfortunately, as we now know, Ikeda was right. The earthquake that gave rise to the devastating 2011 tsunami that devastated Northeast Japan, killing over 18,000 people and causing a meltdown of the nuclear power plant at Fukishima, registered a magnitude of 9.0.

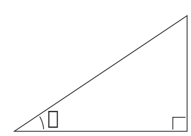
By what factor were the previous seismologists wrong with respect to the energy of the earthquake? (This will take a bit of research. There are good online sources. Earthquake strength is currently measured on the Moment Magnitude Scale, which, like the previous Richter scale, is logarithmic. The energy released by the quake is proportional to its *seismic moment.*) *Fully explain your answer.*

2 Derivatives of the Inverse Trigonometric Functions

1. What is the range of sin? What is the domain of $\sin^{-1}$? Since these agree, we know it is always true that $\sin(\sin^{-1} x) = x$. (It is not true for the composition in the opposite order, since the range of $\sin^{-1}$ does *not* match the domain of sin.)

   Differentiate both sides of $\sin(\sin^{-1} x) = x$ and solve to obtain a formula for $(\sin^{-1})'(x)$.

   Next we will find a much more useful expression for $\cos(\sin^{-1} x)$. To do that, label the triangle at right so that $\sin\theta = x$, and use the Pythagorean Theorem to fill in the length of the remaining side.

   Noting that $-\frac{\pi}{2} \leq \theta = \sin^{-1} x \leq \frac{\pi}{2}$ and that $\cos\theta \geq 0$ in this range, what is another expression for $\cos(\sin^{-1} x)$ that does not involve either cos or $\sin^{-1}$? Use this expression to provide another formula for $(\sin^{-1})'(x)$.

2. Do a similar analysis to derive useful formulas for the derivatives of $\cos^{-1}$ and $\tan^{-1}$. *Fill in all details!*