Tiling of Surfaces with Voronoi Cells

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Abstract

We study the Voronoi cell distributions of points for surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$.The idea is to find point distributions that have Voronoi cells which are either regular hexagons or nearly regular hexagons with a few exceptions.

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0. Introduction

1. Recent work by Saff [1] concerns finding a uniform distribution for some set of points on a surface.

2. The approach is to embed the surface in $\mathbb{R}^3$ (or possibly $\mathbb{R}^4$) and to assign a potential to the distribution, say

$$E = \sum_{i \neq j} \frac{1}{|x_i - x_j|^r},$$

   such that $|x_i - x_j|$ is the distance between points in $\mathbb{R}^3$.

3. By minimizing the potential energy of the system we hope to obtain a fairly uniform distribution.

4. When $s = 1$ the potential is related to the Coulomb potential function.

1. For example, for $r = 1$, the torus will yield a result such that there are virtually no points on the inside portion of the torus.

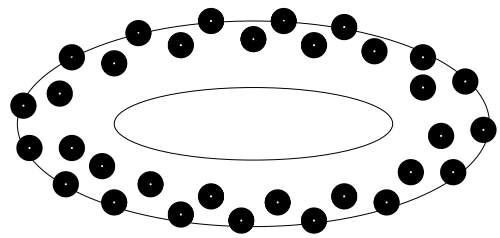

Figure 1. *The points cluster on the outside of the torus .*

2. A reasonable solution occurs as r tends to infinity for these potential functions.
3. As r tends to infinity the potential energy is mostly determined by the relations to the 'nearest neighbor'.
4. the problem can be treated for a variety of cases using simple geometry and redefining our goal in a geometric way.

Observations:

1. For a 'nearest neighbor' potential (think Coulomb which dies rapidly for the distances involved) on a surface M an absolute minimum for some set of n points ought to occur if M can be tiled by n congruent regular hexagons.

2. For a 'nearest neighbor' potential on a surface M a near minimum for some set of n points ought to occur if M can be tiled by n n nearly regular hexagons.

3. Tiling by nearly regular hexagons is equivalent to triangulating the surface by 'nearly equilateral' triangles.

1. Define a point to be a *nearest neighbor* of another point if the two points have a boundary in common.

2. If a surface can be completely tiled by n regular hexagons then there are exactly six nearest neighbors.

3. In this case the distances to the nearest neighbors are equal by congruence and so the sum of the forces on the given point due to the nearest neighbor potentials is zero. If the point is perturbed slightly then the force increases.

4. The hexagonal Voronoi cell is the natural solution by the usual sphere packing arguments.

1. Flat Surfaces

1. Let $S = T^2$ the flat torus, then it is possible to embedd the torus in $\mathbb{R}^4$ Represented by a rectangle in $\mathbb{R}^2$.
2. We can classify these rectangles (up to similarity) by the ratio of their sides.

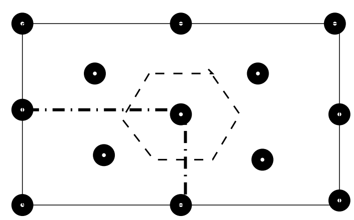

Figure 2. *A basic packing cell and the hexagonal Voronoi cell*

3. A basic packing cell is in ratio 1 to $\sqrt{3}$.
4. Any rectangle constructed using basic packing cells has a regular hexagonal Voronoi cell tiling.
5. This problem is related to the integer lattice problem approximation problem for incommensurable numbers.

1. Let ρ denote the ratio of the height to base for the rectangle representation of the flat torus.

2. For any ratio ρ there exits a point distribution using basic packing cells such that the cells make up a rectangular region of the torus representation in the plane which is within ϵ of the entire area for any ϵ.

3. By slightly stretching the cell packing we will obtain a 'nearly regular' hexagonal Voronoi cell tiling of the torus.

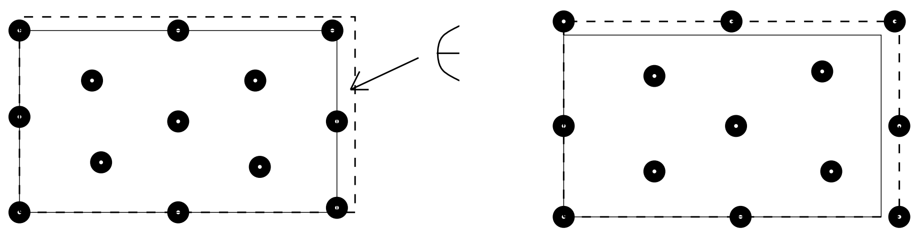

Figure 3. *The tiling is a stretch*

1. The cone is another example of a flat surface.

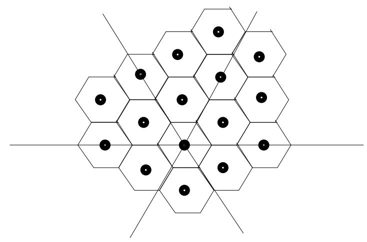

Figure 4. *A regular tiling of the plane applied to cones.*

2. Evidently it is possible to obtain a hexagonal tiling of the flat cone constructed with an angle of $60^o, 120^o, 180^o, 240^o$ and 300^o.

3. These are cones that can be embedded in $\mathbb{R}^3$.

4. These solutions can be rescaled and so one can obtain triangulations a small as you might like.

5. If the cone is constructed from a wedge with angle between 60^o and 65^o one could for instance stretch the 60^o angle and preserve the hexagonal Voronoi cell form though it will not be a regular hexagonal tiling.

2. The Sphere

Proposition 2.1 *There are no tilings of the sphere by regular hexagons*

Outline of Proof:

1. Assume that there is a tiling of the sphere by regular hexagons.
2. This tiling will produce a tiling of the sphere by congruent isosceles triangles.
3. Note that there are tilings of the sphere by equilateral triangles given by the regular solids, but none of these tilings corresponds to a hexagonal tiling.
4. The apex angles for such an isosceles triangle is 60^o. On the other hand the Gauss-Bonnet theorem for the hexagon gives

   $$\frac{A}{2\pi} + \frac{1}{2\pi}\int \kappa_g d\sigma = 1,$$

   where A is the area of the hexagon, κ_g represents the geodesic curvature (which is a discrete sum in this case) and the right hand side represents the Euler characteristic.

Proof Continued:

5. Take the area $A = 4\pi/n$

6. Assume that the integral is $6\alpha/2\pi$ where α represents the exterior angle at each vertex.

7. Then
$$\frac{2}{n} + \frac{3\alpha}{\pi} = 1, \quad \text{or} \quad \alpha > \frac{\pi}{3}.$$

8. This implies that the base angles of the isosceles triangles are greater than 60^o.

9. There must be at least three hexagonal tiles that meet which is a contradiction (i.e. the interior angles must all be 120^o for three congruent tiles to fit at a point).

1. It is well known on the sphere that the ratio of the circumference of a circle to the of that area of that circle is larger than the associated ratio for the flat surface.

2. In order to obtain a near-hexagonal tiling we must cut some of the area out starting in the center as shown below.

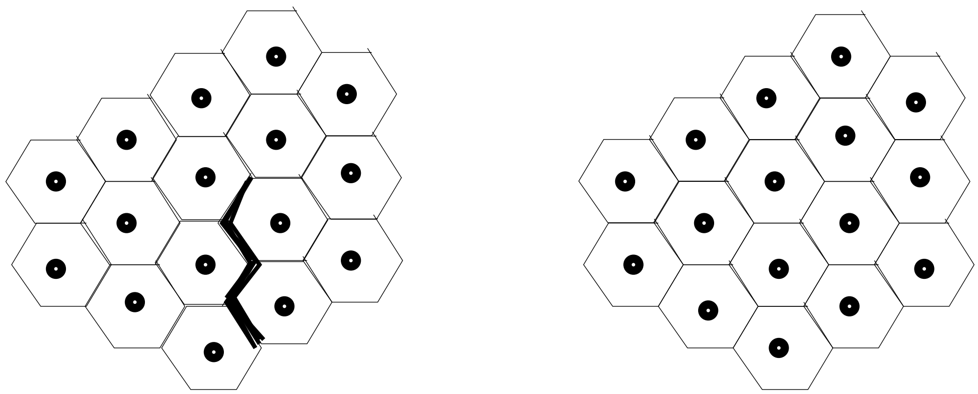

Figure 5. *A near-regular-tiling for a portion of the sphere on the left and a tiling of flat space on the right.*

1. One can construct the isocohedron using 12 copies of a pentagon made from 5 equilateral spherical triangles.

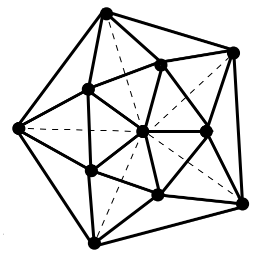

Figure 6. *A portion of isocohedral tiling of the sphere by equilateral triangles.*

2. A near-equilateral triangular tiling of each equilateral triangle in the isocohedron gives a near-regular-tiling of the sphere.
3. Each triangle in the equilateral tiling must have an interior angle 72^o by the Gauss-Bonnet Theorem:
4. $$\frac{4\pi}{20 \cdot 2\pi} + \frac{3\alpha}{2\pi} = 1$$

   implies that $\alpha = 3\pi/5$.

1. For example the regular octahedron coincides with a tiling of the unit sphere by eight triangles.

2. The eight triangles are equilateral and the geodesic curvature of the boundary of such a triangle is isolated in the three corners.

3. Then

$$\alpha = \frac{\pi}{2}$$

which coincides with experience.

4. For the dodecahedron, there are 12 regular pentagons which tile the sphere. Therefore, the area is $A = \pi/3$ and we have

$$\alpha = \frac{5\pi}{15} = \frac{\pi}{3}.$$

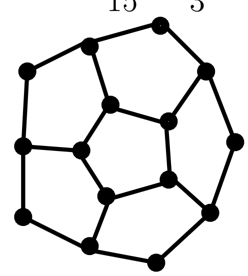

Figure 7. *A portion of the regular pentagonal tiling of the sphere.*

3. Hyperbolic Surfaces

Proposition 3.1 *There are no regular hexagonal honeycomb tilings for the hyperbolic plane (assume $K = -1$ in this case).*

1. The Gauss-Bonnet formula for exterior angles of regular n-polygons is

$$\alpha = \frac{2\pi + A}{n}$$

2. For the hexagon we obtain

$$\alpha = \frac{\pi}{3} + \frac{A}{6} > \frac{\pi}{3}$$

3. Thus the interior angle for a regular hexagon is less that 120^o.

4. A honeycomb tiling requires that three tiles fit at a vertex.

Proposition 3.2 *There exist regular n-polygon tilings for the hyperbolic plane for $n > 6$.*

1. Applying the Gauss-Bonnet formula for an appropriate choice of A (i.e., $A = (n-6)\pi/3$) we have
$$\alpha = \frac{\pi}{3}$$
2. In this case we obtain that the interior angle for a regular n-polygon is 120^o.
3. This implies that exactly three tiles fit at a vertex.
4. By translations and rotations the pattern can be repeated indefinitely.
5. Each regular n-polygon can be divided into n isosceles triangles (not equilateral).

1. For $n = 8$ we have $A = 2\pi/3$.

2. The octagon figure is related to a surface of genus 2.

3. The isosceles triangle formed from the center point to a boundary line of the octagon has base angles equal to 60^o and apex angle 45^o.

4. Each side of this triangle may be cut into k parts and a hexagonal pattern is formed except at the origin and at the identification point for the vertices on the surface of genus 2.

5. The Voronoi cell at the exceptional points is octagonal.

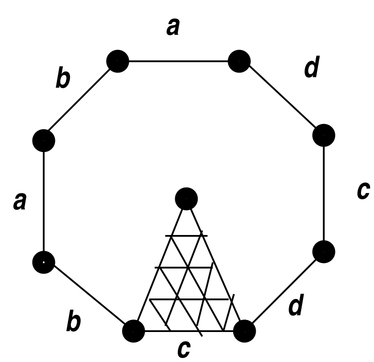

Figure 8. *Exceptional points are in the center and on the boundary.*

1. For $n = 12$ we have $A = 2\pi \approx 6.28$.
2. The dodecagon figure is related to a surface of genus 3.
3. The isosceles triangle formed from the center point to a boundary line of the octagon has base angles equal to 60^o and apex angle 30^o.
4. Each side of this triangle may be cut into k parts and a hexagonal pattern is formed except at the origin and at the identification point for the vertices on the surface of genus 2.
5. The Voronoi cell at the exceptional points is 12 sided.
6. Apparently, the cells are not very close to regular in this case.
7. The number of cells needed near the boundary must increase more rapidly in the hyperbolic case.

4. Surfaces of Revolution

1. The hexagonal numbers are a natural progression for covering the plane with hexagonal Voronoi cells.
2. The progression is $\{1, 6, 12, 18, \ldots\}$.

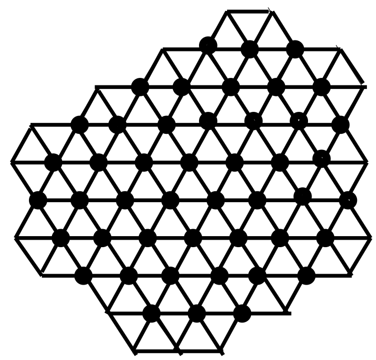

Figure 8. *The hexagonal numbers tiling in the plane.*

3. Any surface can be locally tiled using the hexagonal pattern on the tangent space at a point followed by the exponential projection onto the surface.
4. The problem is patching such local result together.

1. A surface of revolution can be approximated by a series of truncated cones.
2. A hexagonal near tiling can be obtained if a hexagonal progression can be used on the truncated cones so that the Voronoi cells are all hexagonal.
3. The dodecahedron tiling reduced to triangles and the isocahedron tiling of the sphere are examples of this idea.
4. The tilings are not perfect hexagonal progressions.
5. The isocahedron tiling is a $\{1, 5, 5, 1\}$ progression.

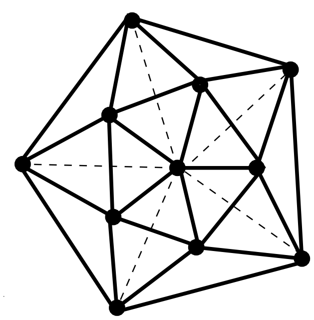

Figure 8. *The isocahedron tiling.*

1. If we divide a surface of revolution into truncated cones, then the boundary circles are not geodesics (in general).

2. Let m represent the slope of the truncated cone, and suppose that n points are distributed evenly on the larger circumference $C = 2\pi r$.

3. Suppose that the distance between the circular boundaries is $\sqrt{3}\pi r/n$, i.e. this corresponds to an equilateral triangle relation if the circumferences are almost the same and there are n points on each circumference.

4. Define $\rho = n\sqrt{1+m^2}/(n\sqrt{1+m^2} - \sqrt{3}\pi m)$.

5. Then ρ represents the ratio between two consecutive progression numbers.

6. Adding a single point between rows of equal points leads to exceptional cells.

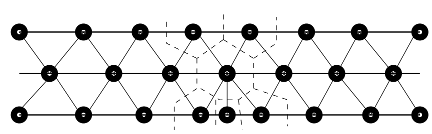

Figure 8. *A five sided Voronoi cell and a seven sided Voronoi cell are required.*

Bibliography

[1] Saff, Edward., *The Poppy-Seed Bagel Theorem*, Invited Lecture at The Mathematics Symposium at Western Kentucky U., Nov. 2005.

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