Calculus For The Biological Sciences

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CHAPTER 1. THE THEORY OF FUNCTIONS

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Chapter 1. The Theory of Functions

1.1 The Cartesian Coordinate System

It is said that Rene Descartes first conceived of the idea of the representing curves on a coordinate axis while staring at a common house fly (Musca Domestica) roaming on his ceiling

The set of *natural numbers*, **N**, written as

$$\{1, 2, 3, \ldots\}$$

is made up from the numbers used for counting. The set of natural numbers are important in all cultures. Different peoples may use different systems for counting. The Mayans used a base 20 system. The Babylonians are known to have used a base 60 system which was favorable for writing fractions (60 has many natural divisors) and possibly since it nearly divides the number of days in a year.

The set of *integers*, **Z**, written as

$$\{\ldots, -3, -2, -1,\ 0,\ 1,\ 2,\ 3, \ldots\}$$

is made up from the set of all natural numbers, their negative values and zero. The set of *rational numbers*, **Q**, is made from the set of all proper quotients of integers (we may not divide by zero) such that any two quotients represent the same number or are equal

$$\frac{p_1}{q_1} = \frac{p_2}{q_2},$$

if and only if

$$p_1 q_2 = p_2 q_1.$$

It is well known from geometry that not every length can be represented by a rational number. For example, $\sqrt{2}$, which is the length of the diagonal of the unit square, is not a rational number as is quickly seen if we attempt to write the prime decomposition of its square as squares of primes.

We may think of the set of *real numbers*, **R**, as the set of all numbers which are the set of all mixed decimal numbers. Given this understanding we may write a real number in the form of a mixed decimal expansion such as

$$2.10203145\ldots,$$

with an integer part, 2, and a decimal part, $0.10203145\ldots$, such that the ellipses indicate other possible values that are not written. The meaning then is the number which is the limit of the sum

$$2+\frac{1}{10}+\frac{0}{10^2}+\frac{2}{10^3}+\frac{0}{10^4}+\frac{3}{10^5}+\frac{1}{10^6}+\frac{4}{10^7}+\frac{5}{10^8}+\ldots,$$

where only a finite number of terms are usually represented in the expansion form.

The real numbers are also represented by the geometric depiction of a straight line with the numbers representing distances from a fixed point referred to as the origin, or zero:

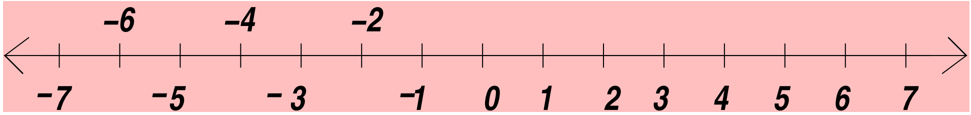


Figure 1.1 *A representation of the real line*

We assume that the reader is familiar with the properties of the real numbers: commutativity of addition and multiplication; associativity of addition and multiplication; distributive properties; inverse properties and so forth. The absolute value of a real number can be defined as the distance of that number from zero. The distance between two points a and b on the real number line is defined as the absolute value of their difference indepedent of ordering:

$$D=|a-b|=|b-a|.$$

We will need the use of intervals throughout. We define the interval types using set notation

$$(a,b)=\{x\in\mathbf{R}|\mathbf{a}<\mathbf{x}<\mathbf{b}\}$$

$$[a,b)=\{x\in\mathbf{R}|\mathbf{a}\leq\mathbf{x}<\mathbf{b}\}$$

$$(a, b] = \{x \in \mathbf{R} | \mathbf{a < x \leq b}\}$$

$$[a, b] = \{x \in \mathbf{R} | \mathbf{a \leq x \leq b}\}$$

We may also define the absolute value expression by the following

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a < 0. \end{cases}$$

We have the following well known properties

Remark 1.1 *For $a, b, r \in \mathbf{R}$,*

1. $|-a| = |a|$.
2. $|a \pm b| \leq |a| + |b|$.
3. $r|a| \leq |r| \cdot |a|$.
4. *If $r > 0$ and $|x - a| < r$, then*

$$-r < x - a < r, \quad \text{or} \quad -r + a < x < r + a.$$

We may represent the geometric plane as pairs of real numbers

$$\{(x, y) \mid x, y \in \mathbf{R} \}.$$

It is natural then to consider a pair of real lines perpendicular to each other which coincide at the point $(0, 0)$ in the geometric plane. It is customary to denote the plane by $\mathbf{R}^2$.

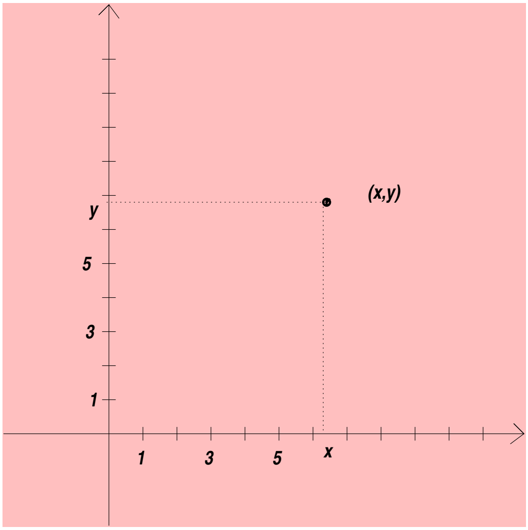


Figure 1.2 *A representation of the coordinate plane*

The distance formula between points (x_1, y_1) and (x_2, y_2) in the plane is given by

$$\boxed{D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

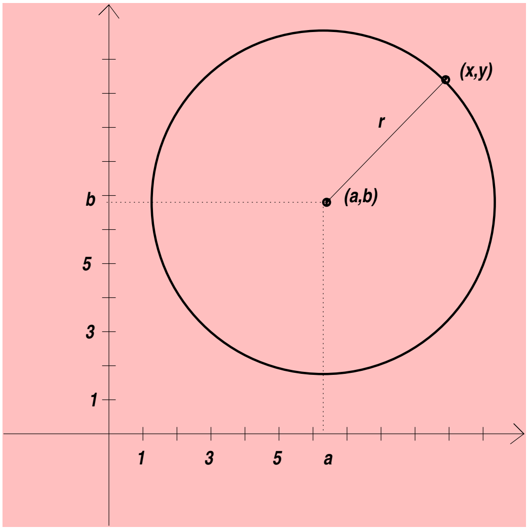


Figure 1.3 *A circle with center (a, b) in the coordinate plane*

Thus the formula for the circle centered at (a, b) of radius r is

$$\boxed{\{(x,y) \in \mathbf{R}^2 \mid (x-a)^2 + (y-b)^2 = r^2\ \}.}$$

The graph of a straight line is determined by either two points in the plane or by one point and the slope of the line with respect to the x-axis. Given the points (x_1, y_1) and (x_2, y_2), we can compute the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The equation of the line can be given in three different forms: The point-slope form; the slope-intercept form; the standard form. The point-slope form is given by

$$y - y_1 = m(x - x_1), \quad or, \quad y - y_2 = m(x - x_2).$$

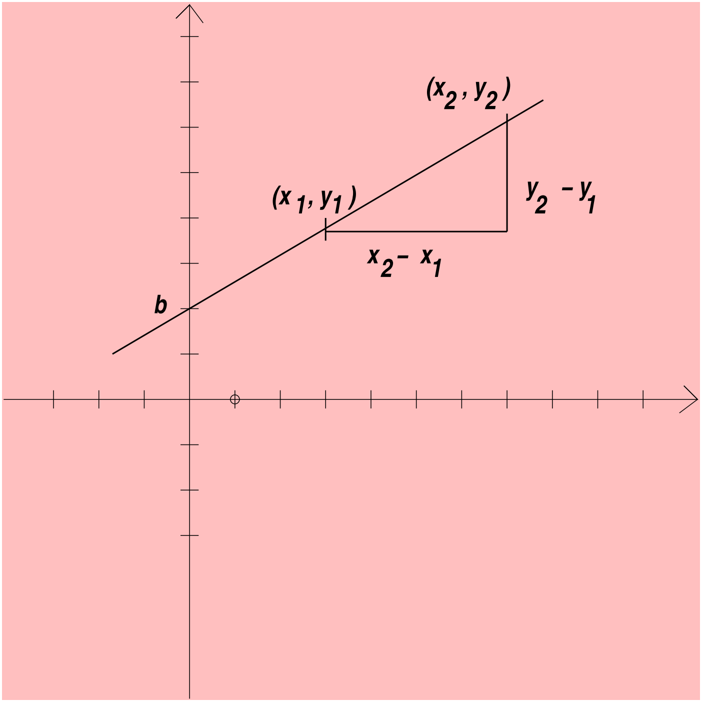


Figure 1.4 *The graph of the equation of a line.*

The slope intercept form is

$$y = mx + b, \quad \text{where} \quad b = y_2 - mx_1.$$

The point $(0, b)$ is the y-intercept. The standard form is

$$\alpha x + \beta y = \gamma,$$

where $m = -\alpha/\beta$ and $b = \gamma/\beta$.

Example 1.1 The points $(1, 1)$ and $(2, -1)$ determine a straight line. Write the equation of the straight line in the point-slope form, slope-intercept form, and the standard form. Graph the straight line.

Solution 1.1 Let $(x_1, y_1) = (1, 1)$. Then the slope is

$$m = \frac{-1 - 1}{2 - 1} = -2.$$

The point-slope form is

$$y - 1 = -2(x - 1).$$

The slope-intercept form is

$$y = -2x + 2 + 1, \quad \text{or} \quad y = -2x + 3.$$

The standard form may be wriiten as

$$2x + y = 5.$$

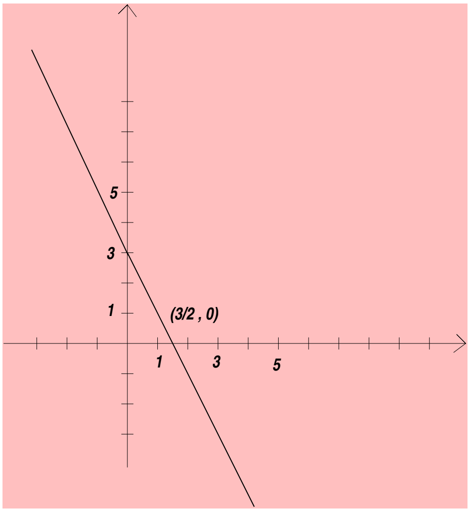


Figure 1.5 *The graph of the equation of a line.*

A quadratic equation has the form

$$y = ax^2 + bx + c, \quad a, b, c \in \mathbf{R}.$$

If $a > 0$ then the shape of the graph is upward. The is a minimum point which is called the vertex. The graph has a mirror symmetry about the vertical line through the vertex. If the graph touches the x-axis the point or points at which the graph touches are called roots of the graph. The roots of the graph of a quadratic equation occur at equal distances from the vertical line as in the example below.

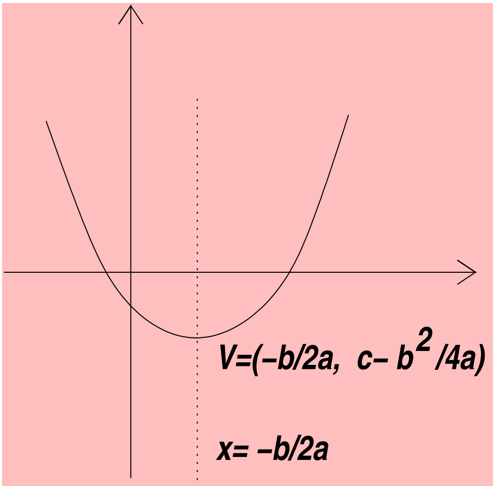


Figure 1.6 *The graph of a quadratic equation with the vertical line of symmetry.*

The quadratic formula gives the value of the vertex on the x-axis and the roots if they exist.

The Quadratic Formula: The roots of $y = ax^2 + bx + c$ are given by

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The line of symmetry is $x = -b/2a$ and the vertex is $V = (-b/2a, c - b^2/4a)$.

Example 1.2 Find the line of symmetry and the vertex of the quadratic equation $y = x^2 - 6x + 9$ and graph it.

Solution 1.2 According to the quadratic formula we have the following: $a = 1; b = -2; c = 1$. The line of symmetry is given by

$$x = \frac{-b}{2a} = \frac{2}{2}, \quad \text{or,} \quad x = 1.$$

The vertex is at the point $(1, 1^2 - 2 \cdot 1 + 1) = (1, 0)$. The roots are at

$$x = 1 \pm \frac{\sqrt{4-4}}{2} = 1.$$

This is called a double roots and indicates that the graph touches at the x-axis at the vertex but does not cross the axis.

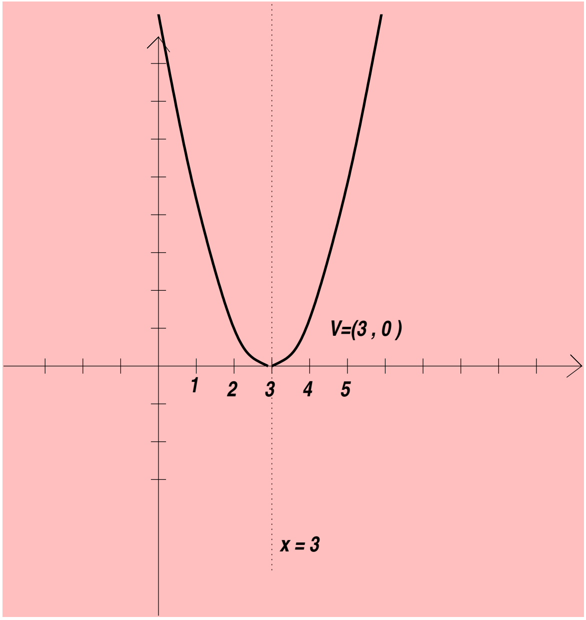


Figure 1.7 *The graph of the equation of a line.*

Example 1.3 Find the line of symmetry and the vertex of the quadratic equation $y = x^2 + 2x - 8$ and graph it.

Solution 1.3 According to the quadratic formula we have the following: $a = 1; b = 2; c = -8$. The line of symmetry is given by

$$x = \frac{-b}{2a} = \frac{-2}{2}, \quad \text{or}, \quad x = -1.$$

The vertex is at the point $(-1, (-1)^2 + 2 \cdot (-1) - 8) = (-1, -9)$. The roots are at

$$x = -1 \pm \frac{\sqrt{4+32}}{2} = -1 \pm 3, \quad \text{or}, \quad x = -4, 2.$$

The graph crosses at the points $(-4, 0)$ and $(2, 0)$.

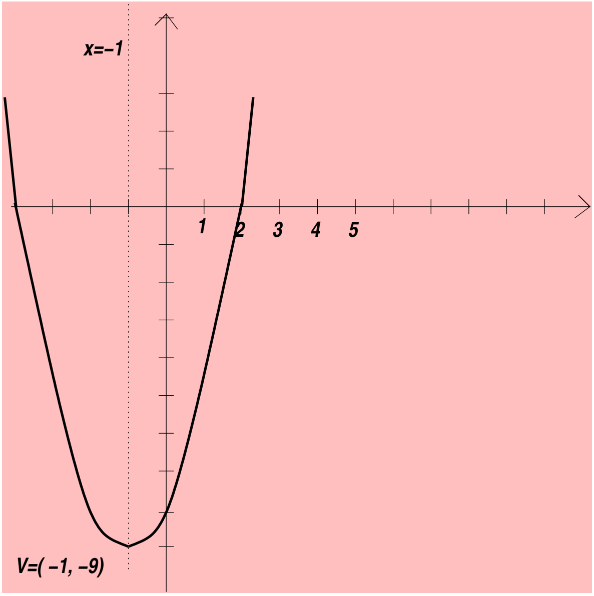


Figure 1.8 *The graph of the equation of a line.*

Exercises 1.1

A. Describe the interval associated to the given set.

1. $\{x \in \mathbf{R} \mid -1 < x \leq 0\}$
2. $\{x \in \mathbf{R} \mid 1 < x < 3\}$
3. $\{x \in \mathbf{R} \mid -1 \leq x \leq 1\}$

4. $\{x \in \mathbf{R} \mid -2 \leq x < -1\}$
5. $\{x \in \mathbf{R} \mid x \leq 0\}$
6. $\{x \in \mathbf{R} \mid -5 < x\}$
7. $\{x \in \mathbf{R} \mid -1 < x \leq 1\} \cap \{x \in \mathbf{R} \mid 0 \leq x < 2\}$

B. Describe the interval in set notation.

1. $[0, 1]$
2. $(-1, 2]$
3. $(-\infty, 3]$
4. $[0, \infty)$
5. $(-2, 0)$
6. $[-1, 1] \cap (0, 3)$
7. $[-2, \infty) \cap (-4, 1]$

C. Evaluate the expressions.

1. $|-1|$
2. $|2-4|$
3. $|3-1|$
4. $|(-2)^2|$
5. $|(-2)(-3)|$

D. Simplify the following.

1. $|2+x-1|$
2. $|x^2+2x+1|$
3. $|-3(x-1)|$
4. $|x^3+3x^2+3x+1|$

E. Plot the ordered pairs in the cartesian coordinate system.

1. $(0,1); (0,0); (1,1), (-1,-1)$.
2. $(0,0); (2,0); (1,2)$.
3. $(0,1); (0,0); (1,0), (1,1)$
4. $(0,-1); (1,0); (2,1); (3,2)$.

F. Compute the distance between points.

1. $(2,3); (-1,5)$.
2. $(2,0); (1,2)$.
3. $(0,0); (1,1)$.
4. $(5,5); (-1,-1)$.

G. Graph the circle described.

1. A circle with center $(1,1)$ and radius 2.
2. A circle with center $(1,0)$ passing through $(0,0)$.
3. A circle with center $(-1,-1)$ passing through $(1,1)$.
4. A circle with diameter given by points $(1,1)$ and $(3,3)$.

H. Graph the following.

1. Graph the equation of the line through the points $(1,2)$, and $(2,-1)$.
2. Graph the equation of the line through the point with slope $m = 2$.
3. Graph the equation of the line through $(1,1)$ perpendicular to the line $y = x + 2$.

I. Graph the following.

1. Find the line of symmetry and the vertex of the quadratic equation $y = x^2 - 2x + 1$ and graph it.

2. Find the line of symmetry and the vertex of the quadratic equation $y = x^2 - 4x + 4$ and graph it.
3. Find the line of symmetry and the vertex of the quadratic equation $y = x^2 + 1$ and graph it.
4. Find the line of symmetry and the vertex of the quadratic equation $y = -x^2 + 1$ and graph it.

1.2 The Definition of a Real Valued Function

The notion of a function is very old and is apparent in the work of geometers such as Rene Descartes and Issac Newton though the first use of the term is attributed to the famous philosopher mathematician Gottfried Leibniz. Functions were first used to study curves in space and in particular the rates of change of particles in motion. Galileo Galilei was able to estimate the acceleration of gravity by observing the periodic pendulum motion of a hanging lamp. Hence the famous deduction that all objects fall at the same rate of acceleration.

Functions are used to estimate or model relations between measured variables or values. For instance, one can calculate the volume V or the surface area S of a sphere by measuring the diameter x of the sphere:

$$V = \frac{\pi x^3}{6}, \quad S = \pi x^2, \quad x \geq 0.$$

Definition 2.1 A *real valued function* f with doamin $D \subset \mathbf{R}$ and target $T \subset \mathbf{R}$ is a unique assignment $f(x) \in T$ for each each $x \in D$. We write

$$f : D \to T$$

to denote this assignment.

The Idea of a Function: A *real valued function* is a relation (or set of ordered pairs) between an independent real variable, say x, in the domain D and a dependent real variable, say y, such that exactly one y is related to each x.

Definition. Assume that $f(x)$ and $g(x)$ are real valued functions defined on a common domain. Then

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(f \cdot g)(x) = f(x)g(x)$
4. $(f/g)(x) = f(x)/g(x)$, when $g(x) \neq 0$

Definition. Assume that $f(x)$ and $g(x)$ are real valued functions such that the domain of $g(x)$ is in the range of $f(x)$. Then we define the *composition function*

$$f \circ g(x) = f(g(x)).$$

Whenever we have the composition $f(g(x))$ we will refer to $g(x)$ as the inside function for the composition and $f(x)$ as the outside function. Most of the functions we will study are defined by the rule method, that is, a rule is given to compute the value of the function for each x in the domain. The variable x in the rule format is called the argument of the function. Examples of rule definitions are given by

$$f(x) = \sqrt{x}, \quad \text{and} \quad g(x) = x^2 + 1.$$

If we wish to know the value of $f(x)$ at $x = 2$ we obtain

$$f(2) = \sqrt{2}.$$

If we wish to know the value of the function $g(x)$ at $x = -1$ we have

$$g(-1) = (-1)^2 + 1 = 2.$$

Using the definitions above we can define a function in terms of a rule using sums, products, quotients and compositions of simple functions which are also defined by rules.

Example 2.1 Suppose that $g(x) = x^2 + 1$ and $f(x) = sqrt(x)$. Find $f \circ g$ and $g \circ f$.

Solution 2.1 Observe that the domain of $g(x)$ is $D_g\mathbf{R}$ and the range is $R_g = \{x \geq 1\}$. The domain of $f(x)$ is $D_f = \{x > 0\}$ and the range is

$\{x \geq 0\}$. Using these facts we have that domain of $f \circ g$ is $D_{f \circ g} = \mathbf{R}$ since the range of $g(x)$ is inside the domain of $f(x)$ and the domain of $g(x)$ is $\mathbf{R}$. Now we can write the function

$$f \circ g(x) = f(x^2 + 1) = \sqrt{x^2 + 1}, x \in \mathbf{R}.$$

On the other hand, the domain of $g \circ f$ is $D_{g \circ f} = \{x \geq 0\}$ and we have

$$g \circ f(x) = g(\sqrt{x}) = (\sqrt{x})^2 + 1, \quad x \geq 0.$$

In the case of the relation $V = \pi x^3/6$ we have that the volume of the sphere is a function of the diameter. We say that the diameter is an independent variable and the volume is a dependent variable. In particular, when we conceive of this relation as a function we implicitly assume that we can calculate the value of the volume only when we know the value of the diameter. Of course, it is sometimes possible to reverse the relation so that the independent variable and the dependent variable are interchanged. In this case we have that the diameter is given by

$$x = \Big(6V\Big)^{1/3}, \quad V \geq 0.$$

The domain in this case is determined by the inequality $V \geq 0$, since we have taken into account the physical constraints in the problem.

Now observe that if we define the relation

$$f(x) = \frac{V(x)}{S(x)} = \frac{x}{6}, \quad x > 0,$$

then $f(x)$ defines a new function. In particular $f(x)$ is a *linear function.* This means that the relation or set of ordered pairs in the xy-plane

$$\{(x, y) | y = f(x)\}$$

is a straight line graph.

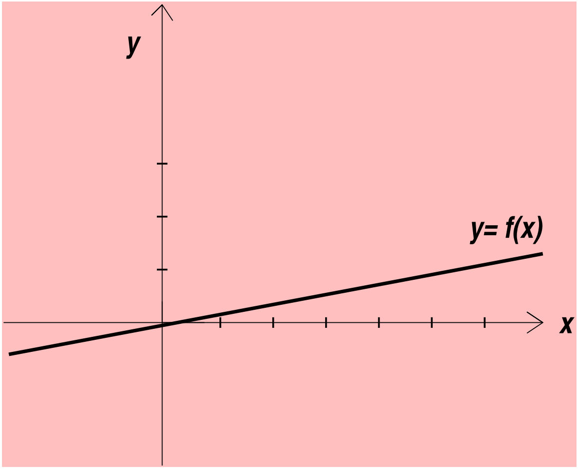


Figure 1.9 *The graph of a linear function*

Now suppose that you can model the volume of an egg using an ellipsoid which is symmetric with respect to a central axis. Let x denote the long diameter of the egg. Suppose that the circular diameter of the egg is $9x/10$, then the volume is given by the function

$$V(x) = \frac{9^2 \pi x^3}{6 \times 10^2}.$$

Example 2.2 The circumference of a circle is in direct proportion to its radius. Give the linear function that represents this proportion.

Solution 2.2 From elementary geometry we have the proportion relation

$$\frac{C}{r} = 2\pi, .$$

This implies the linear function

$$C(r) = 2\pi r, \quad r \geq 0..$$

Simple proportion relations between variables such as in the case above invariably lead to linear function relations.

Example 2.3 Assume the diameter of a tree is in direct proportion to the number of years elapsed beginning at year five. Assume that the diameter in the fifth year is 2 inches and that the proportion constant is 3. Give the linear function that represents this proportion relation.

Solution 2.3 Assume that the diameter is given by D and the time elapsed by t. The assumption allows the proportion

$$\frac{D-2}{t-5} = 3, \quad \text{or} \quad D - 2 = 3t - 15, t \geq 5.$$

This implies the linear function relation

$$D(t) = 3t - 13, \quad t \geq 5.$$

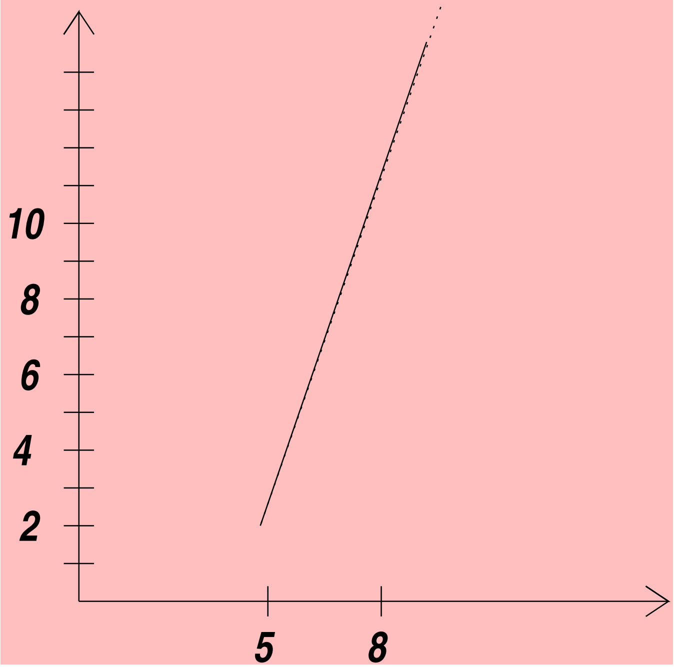


Figure 1.10 *The graph of a linear function*

Definition. Assume that $h(x)$ and $g(x)$ are real valued functions defined on the domains given below. A function may be defined by a split rule as follows

$$f(x) = \begin{cases} g(x), & a \leq x \\ h(x), & x < a. \end{cases}$$

In fact a function can be defined by any number of rules such that the domain is partitioned into disjoint sets and there is a rule given for the function on each of the sets.

Definition. The absolute value function is defined by the rule

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

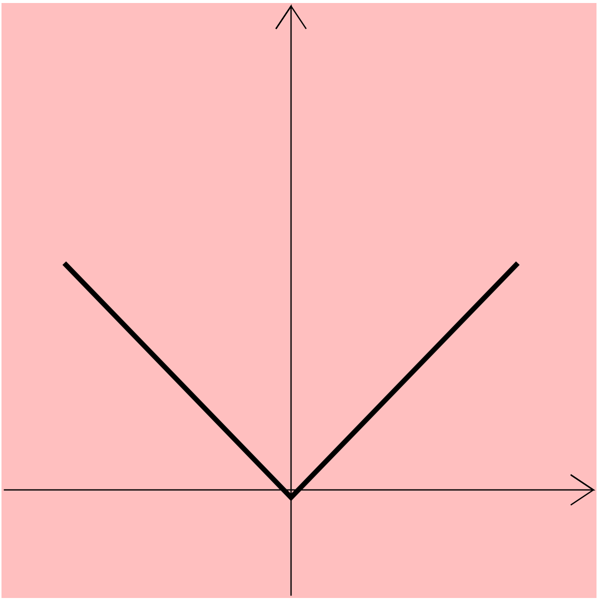


Figure 1.11 *The graph of the absolute value function*

Definition. Assume that $0 \leq r < 1$ and $n \in \mathbf{Z}$. Then the greatest integer function is defined by the rule

$$[[x]] = \begin{cases} n, & x \geq 0 \text{ and } x = n + r \\ n, & x < 0 \text{ and } x = n + r. \end{cases}$$

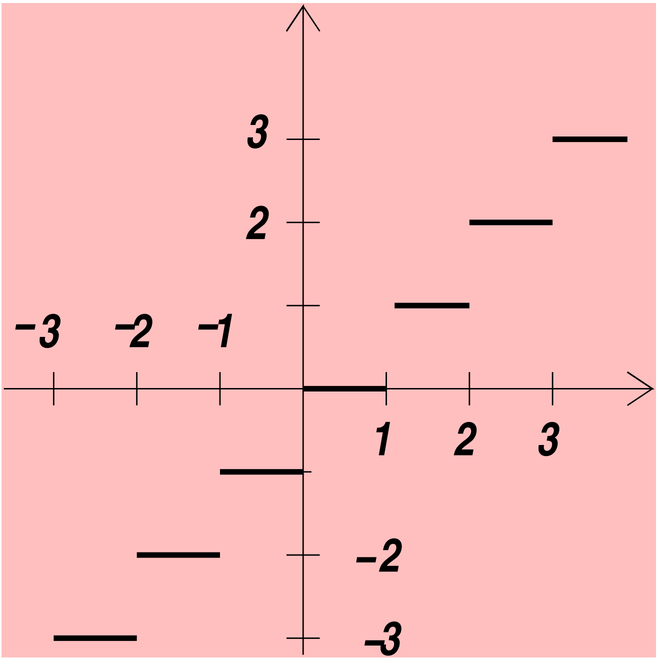
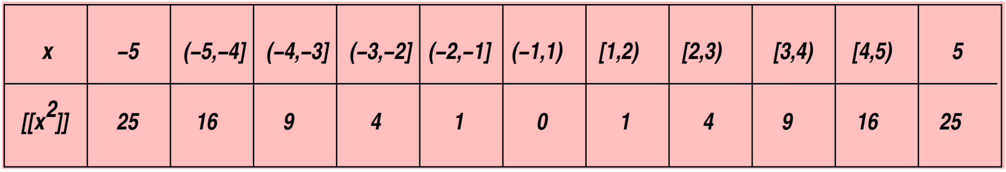


Figure 1.12 *The graph of the greatest integer function*

Example 2.4 Graph the step function defined by $f(x) = [[x^2]]$ on the interval $[-5, 5]$.

Solution 2.4 The following table defines the function in the domain:



x	−5	(−5,−4]	(−4,−3]	(−3,−2]	(−2,−1]	(−1,1)	[1,2)	[2,3)	[3,4)	[4,5)	5
$[[x^2]]$	25	16	9	4	1	0	1	4	9	16	25

Table 1.1

The values are easily graphed:

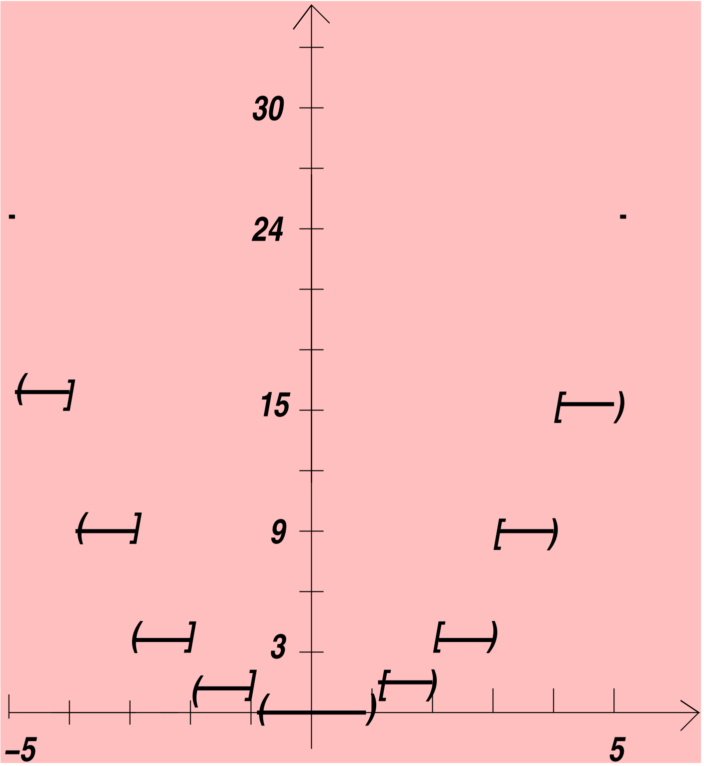


Figure 1.13 *The graph of the composition step function* $f(x) = [[x^2]]$.

In this graph the open parentheses indicate that the end point is not on the graph and the closed parentheses indicate that the end point is on the graph.

Exercises 2.1

A. Assume that $f(x) = x^2$**;** $g(x) = \sqrt{x+1}$**;** $h(x) = x - 1$**. Compute the rule for the following functions.**

1. $(f + g)(x)$
2. $(f \circ g)(x)$
3. $(f - h)(x)$
4. $(g \circ h)(x)$
5. $(f \cdot g)(x)$
6. $(g/h)(x)$

7. $(f \cdot h/g)(x)$

B. The given function is a composition of two functions. Find an inside function $g(x)$ and an outside function $f(x)$ in each case and show that the given function is $f(g(x)$.

1. $h(x) = \sqrt{x^3 + 1}$
2. $h(x) = (x + 1)^7$
3. $h(x) = |x + 1|$
4. $h(x) = [[x^2 + 1]]$
5. $h(x) = (x^2 + 2)^{1/3}$

C. Graph the linear function in each case.

1. $f(x) = x + 1$.
2. $f(x) = 2x - 1$.
3. $f(x) = 1 - 3x$.
4. $f(x) = -x - 1$.
5. $f(x) = 3x + 3$.

1.3 Real Valued Polynomial Functions

The constant function

$$f(x) = a,$$

and the linear function and quadratic functions,

$$g(x) = ax + b, \quad h(x) = ax^2 + bx + c, \quad a, b, c \in \mathbf{R},$$

are examples of polynomial functions. A general real valued *polynomial function* of degree n with 1 variable takes the form

$$\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o,}$$

where $a_n, a_{n-1}, \ldots, a_1, a_o \in \mathbf{R}$ are called the coefficients of the polynomial.

A quadartic function has the general form

$$f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbf{R}.$$

The graph $y = ax^2 + bx + c$ is a parabola. There are two shapes for graphs of quadratic functions. If $a > 0$, then the graph is said to be concave up and has an absolute minimum at some point. The graph is symmetric with respect to the vertical line through the minimum point.

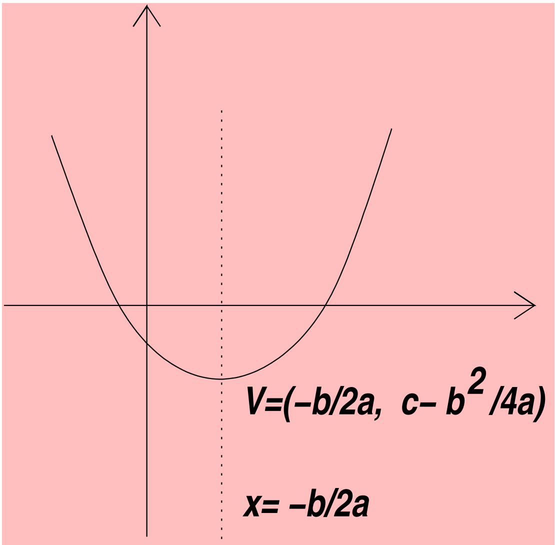


Figure 1.14 *The graph of a quadratic function with a minimum*

If $a < 0$, then the graph is said to be concave down and has an absolute maximum at some point.

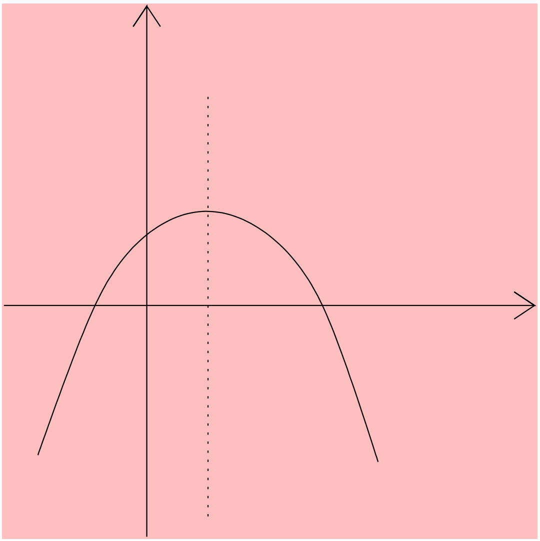


Figure 1.15 *The graph of a quadratic function with a maximum*

Real Distinct Quadratic Roots: *Assume that* $f(x) = ax^2 + bx + c$ *and that* $b^2 - 4ac$*¿ 0. Then the two real roots are equidistant from* $-b/2a$ *at*

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}, \quad \text{and} \quad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

1. *If* $a > 0$*, then the minimum is*

$$f(-b/2a) = \frac{b^2}{4a} - \frac{b^2}{2a} + c = c - \frac{b^2}{4a}.$$

2. *If* $a < 0$*, then the maximum is*

$$f(-b/2a) = c - \frac{b^2}{4a}.$$

3. *The point* $(-b/2a, c - b^2/4a)$ *is called the vertex of the graph of the parabola representing the quadratic function.*

If

an object is thrown directly upward and air ressistance is assumed to be neglegent, then the motion of the objects height is a quadratic function of time.

The Kinematic Formula: *If the acceleration due to gravity is g, the initial velocity is v_o and the initial height is h_o, then the kinematic formula for the height is given by*

$$h(t) = -\frac{g}{2}t^2 + v_o t + h_o,$$
$$v(t) = -gt + v_o.$$

Example 3.1 An archer fish (Toxotes chatareus) shoots a projectile of water at a fly and knocks it into the water. If the archer fish hits the fly at exactly 1 meter directly above it, estimate the minimum initial velocity of the water projectile.

Solution 3.1 The acceleration due to gravity in the metric system is taken as $g = 9.8$ m/sec. Therefore, the kinematic formula for the height of a projectile in meters as a function of time is given by the quadratic function

$$h(t) = -4.9t^2 + v_o t = (v_o - 4.9t)t,$$

where v_o is the unknown initial velocity of the water projectile. This quadratic has roots at $t = 0$ and at $t = v_o/4.9$. It is evident that the maximum height occurs at $t = v_o/9.8$. Thus

$$h(\frac{v_o}{9.8}) = (v_o - \frac{4.9v_o}{9.8})\frac{v_o}{9.8},$$

and

$$h(\frac{v_o}{9.8}) = \frac{v_o^2}{19.6}.$$

But we know that the maximum height of the projectile is at least 1 meter. Therefore,

$$1 = \frac{v_o^2}{19.6}, \quad \text{or,} \quad v_o = \sqrt{19.6} \approx 4.42\text{m/sec}.$$

Example 3.2 The peregrine falcon (Falco peregrinus) can dive, or stoop, at speeds up to 390 kilometers per hour. Assume that the stoop is nearly a free fall dive. In other word assume the kinematic formula. How long will it take to reach this maximum speed and what total distance will be covered in the stoop.

Solution 3.2 The velocity equation in the vertical kinematic formula in meter and seconds is

$$v(t) = -9.8t + v_o.$$

We may assume that the initial vertical velocity is zero. Converting to meters we obtain

$$\frac{390000\text{m}}{3600\text{sec}} = 108\text{m/sec}.$$

Thus

$$108 = 9.8t, \quad \text{or,} \quad t = \frac{108}{9.8}\text{ sec} \approx 11.02\text{ sec}.$$

The distance formula is

$$h(t) = -4.9t^2 \approx 595\text{ m}.$$

Example 3.3 An osprey (Pandion haliaetus) is primarily a fish eater and may sometimes hover as much as 40 meters above water before diving for prey. Assuming no air resistance and that bird undergoes a free fall dive, what is the maximum speed that the osprey can attain in a dive and how much time is required to dive.

Solution 3.3 The kinematic formula gives the function

$$h(t) = -4.9t^2 + 40,$$

where 40 represents the height of the bird above the water. In this function the water is consequently at height $h = 0$ and so the kinematic formula gives the relation

$$0 = -4.9t^2 + 40, \quad \text{or,} \quad t = \sqrt{\frac{40}{4.9}} \approx 2.85\text{ sec}.$$

The velocity at this time is

$$v(2.85) \approx -9.8 \times 2.85 \approx -27.93\text{ m/sec}.$$

The negative sign indicates that the motion is downward.

For the consideration of higher degree polynomials we will define the general form but we will begin by only considering those function whose coefficient of highest degree is 1 or -1.

A polynomial $p(x)$ has a root at some point b if $p(b) = 0$. The cubic polynomial function

$$\boxed{p(x) = a_3x^3 + a_2x^2 + a_1x + a_o, \quad a_o, a_1, a_2, a_3 \in \mathbf{R},}$$

may have three possible distinct roots or it may have fewer. A cubic always has at least one real root since it must always cross the x-axis.

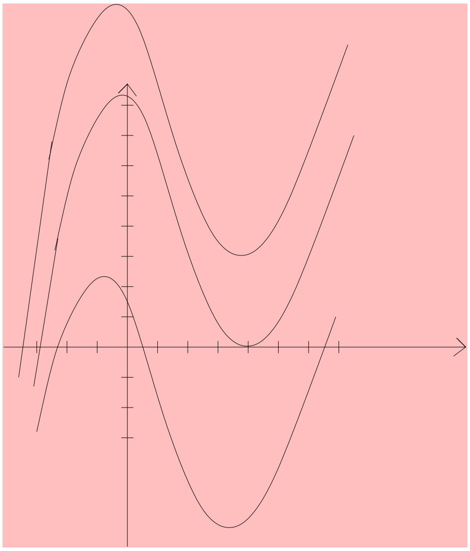


Figure 1.16 *A cubic polynomial with three distinct roots; two roots; one root.*

The most general form of a cubic will have one local maximum and one local minimum, but it may also have the form of a strictly increasing or decreasing function.

To graph a cubic function:

1. Calculate the function at convenient points.
2. Determine the sign of the coefficient of x^3 and use this to determine whether the function tends to infinity as x tends to infinity.
3. Determine the roots if possible and graph the points. If there are three roots you may be able to estimate the local maximum and minimum.
4. If the cubic is of the form $f(x) = (x-a)^3 - b$ then use the graph of $g(x) = x^3$ centered at (a, b).

Example 3.4 Graph the function $f(x) = x^3$.

Solution 3.4 $f(0) = 0$ is the only root of the function. The cubic is an increasing function and so it tends to infinity as x tends to infinity and it tends to negative infinity when x tends to negative infinity.

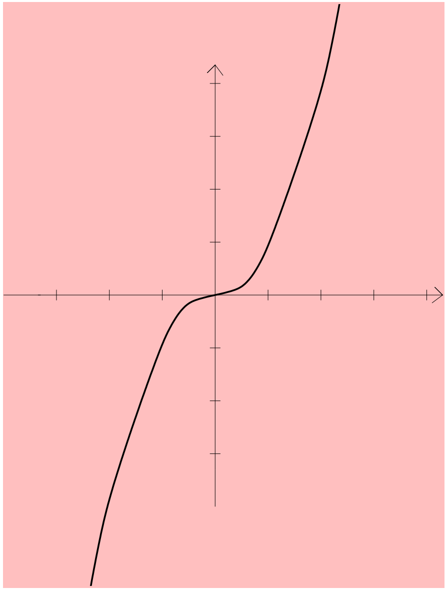


Figure 1.17 *The simple cubic polynomial* $f(x) = x^3$

Example 3.5 Graph the function $f(x) = x^3 + 3x^2 + 3x$.

Solution 3.5 Observe that $f(x) = (x+1)^3 - 1$. We sketch the graph by moving the graph of $y = x^3$ so that it is centered at $(-1,-1)$. We note that $x = 0$ is a root.

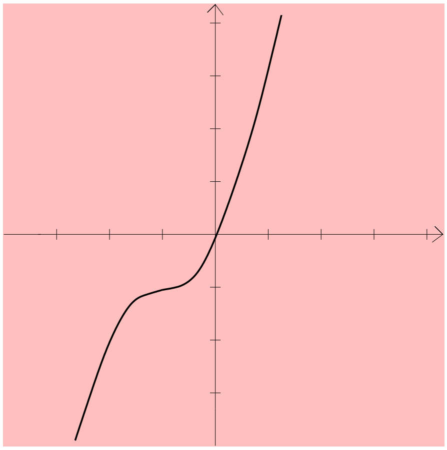


Figure 1.18 *The cubic polynomial* $f(x) = (x+1)^3 - 1$

Example 3.6 Graph the function $f(x) = x^3 + 3x^2 + 3x$.

Solution 3.6 Observe that $f(x) = (x+1)^3 - 1$. We sketch the graph by moving the graph of $y = x^3$ so that it is centered at $(-1,-1)$. We note that $x = 0$ is a root.

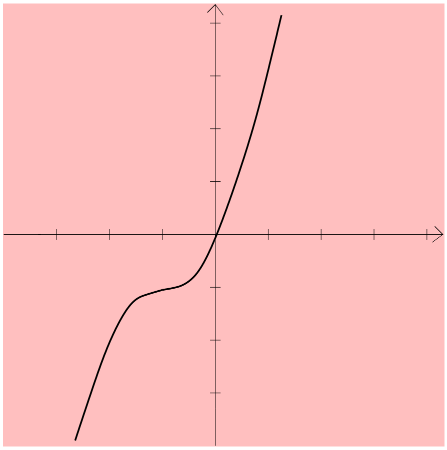


Figure 1.19 *The graph of cubic polynomial* $f(x) = (x+1)^3 - 1$

Example 3.7 Graph the function $f(x) = -x^3 + x$.

Solution 3.7 Note that the function factors as

$$f(x) = x(1 - x^2),$$

and so has roots $x = 0,\ x = 1,\ x = -1$. Since the coefficient of x^3 is $a_3 = -1$, we know that the function tends to infinity as x tends to negative infinity and that the function trends to negative infinity as x tends to positive infinity. We expect a local maximum between $x = 0$ and $x = 1$ and a local minimum between $x = -1$ and $x = 0$. Later in the course, methods of calculus will allow us to determine the exact maximum value.

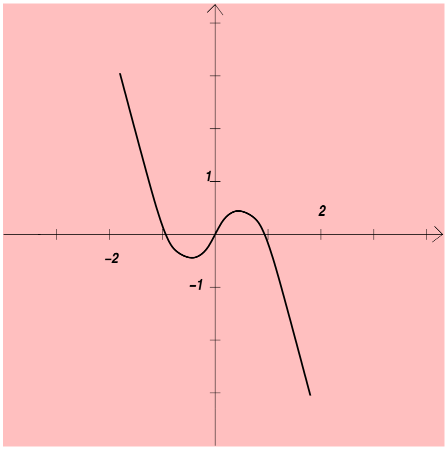


Figure 1.20 *The graph of the cubic polynomial* $f(x) = -x^3 + x$

A degree four polynomial function or a quartic function is given in the form.

$$\boxed{p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \quad a_o, a_1, a_2, a_3, a_4 \in \mathbf{R},}$$

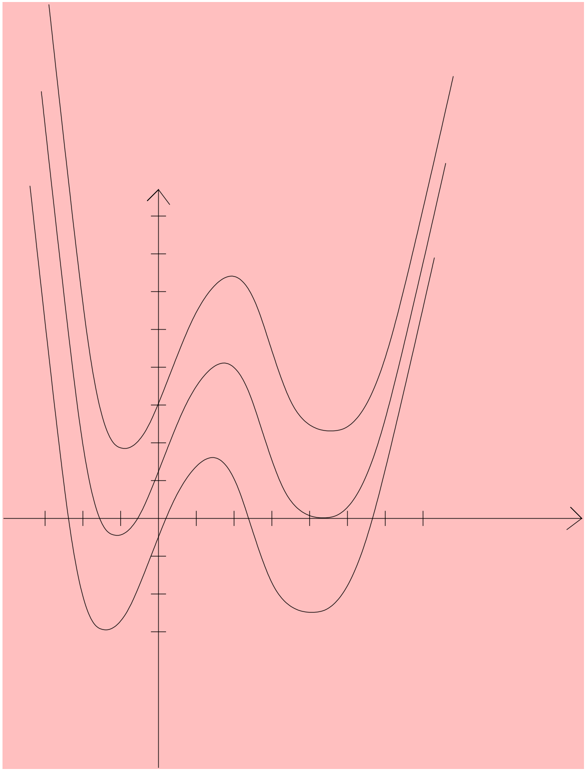


Figure 1.21 *A quartic polynomial with four distinct roots; three roots; no root.*

may have four distinct roots. This fourth degree polynomial may have no real roots. The most general form of a fourth degree polynomial may have three local extrema, that is two local maximums and one local minimum or two local minima and one local maximum. In general a polynomial may have as many distinct real roots as its degree.

Exercize 1.3

A. Sketch the graph of the quadratic functions

1. $f(x) = -x^2 + 2x - 1$.
2. $f(x) = 2x^2 + 4x + 2$.
3. $f(x) = x^2 - 4$.

4. $f(x) = -x^2 + 1$.

B. Sketch the graph of the cubic functions

1. $f(x) = -x^3$.
2. $f(x) = x^3 - x$.
3. $f(x) = x^3 + 1$.
4. $f(x) = (x - 1)^3$.

C. Solve the following problems.

1. An archer fish shoots a projectile of water at a fly and knocks it into the water. If the archer fish hits the fly at exactly 0.25 meters directly above it, estimate the minimum initial velocity of the water projectile.

2. An archer fish shoots a projectile of water at a fly and knocks it into the water. If the fish hits the fly at exactly 1.2 meters directly above it, estimate the minimum initial velocity of the waterprojectile.

3. A peregrine falcon stoops from a height of 500 meters above the prey. Assume that the stoop is nearly a free fall dive. What is the maximum speed and what is the approximate time of the stoop.

4. An osprey hovers 20 meters above water before diving for prey. Assuming no air resistance and that bird undergoes a free fall dive, what is the maximum speed that the osprey can attain in a dive and how much time is required to dive.

5. An osprey dives in a free fall assuming no air resistance and hits the water 4 seconds later. What was the height of the stoop.

1.4 Real Valued Rational Functions

A real valued *rational function* of 1 variable is determined by the function rule

$$\boxed{h(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0,}$$

where $p(x)$ and $q(x)$ are real valued polynomial functions.

We will begin our study of rational functions by considering the rational function defined by the rule.

$$f(x) = \frac{1}{x}, \quad x \neq 0.$$

To graph this function it is necessary to analyze the properties of the function. We observe that the domain is $\mathbf{R} - \{\mathbf{0}\}$ since the denominator has a root at $x = 0$ and the function is undefined there. In addition, the function is always positive when x is positive and negative when x is negative. When $|x|$ is large the value of the function is equally small. When $|x|$ is small the absolute value of the function $|f(x)|$ is equally large. Thus we have the graph

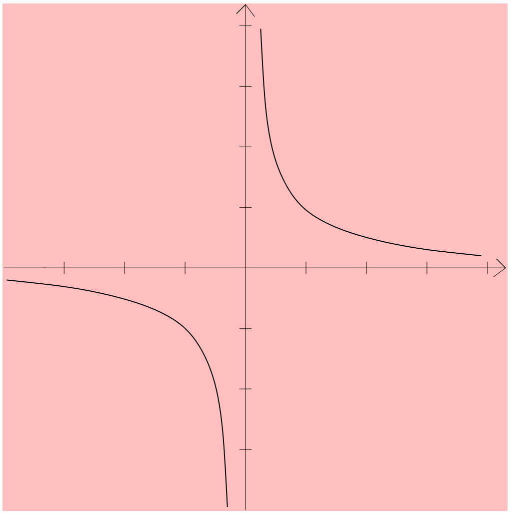


Figure 1.22 *The graph of* $f(x) = \frac{1}{x}$.

The point $x = 0$ is called a singularity for the graph and for the function.

Now consider the function $g(x) = 1/(x-1)$. The graph will have the same properties except that the singularity will be at $x = 1$. We may graph this function by translating the graph of $f(x) = 1/x$ by one unit to the right.

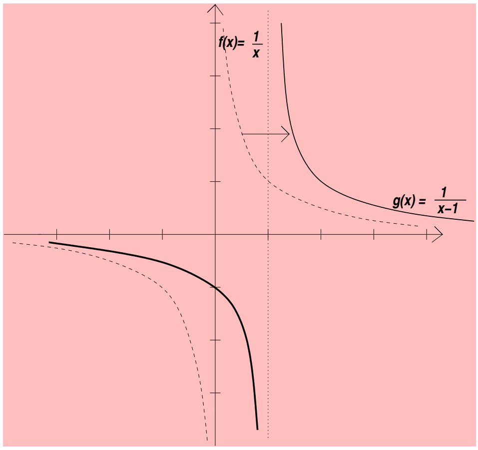


Figure 1.23 *The graph of* $f(x) = \frac{x}{(x-1)}$.

We will refer to the two polynomials that define a rational function as the numerator polynomial and the denominator polynomial. The domain of a rational function is $\mathbf{R}$ if the denominator polynomial has no roots. If the denominator polynomial has roots at $b_1, b_2, b_3, \ldots, b_k$, then the domain is $\mathbf{R} - \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3} \ldots, \mathbf{b_k}\}$ whether or not any of these roots apear in the numerator polynomial.

To graph a rational function we must determine the roots of the numerator polynomial and the denominator polynomial. Next we may simplify the rational function by cancelling like factors in the numerator and the denominator polynomials. For instance, if

$$f(x) = \frac{x^2 + 2x}{x^2 + x - 2}$$

then the domain is $\mathbf{R} - \{\mathbf{1}, \mathbf{2}\}$. The function simplifies to

$$f(x) = \frac{(x+2)x}{(x+2)(x-1)} = \frac{x}{x-1}, \quad x \neq 2.$$

Since $x = 0$ is a root of odd degree, the function will change sign at the numerator root $x = 0$. Since $x - 1$ is a root of odd degree, the function will tend to infinity near the denominator root $x = 1$ and will change sign across this point. Finally, the function is not defined at $x = -2$ since this point is not in the domain. We can simplify this by writing

$$f(x) = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}, \quad x \neq 2.$$

The graph is similar to the previous graph except that the function is undefined at $x = 2$. The circle in the graph indicates that the point $(2, 2)$ is missing from the graph. We have

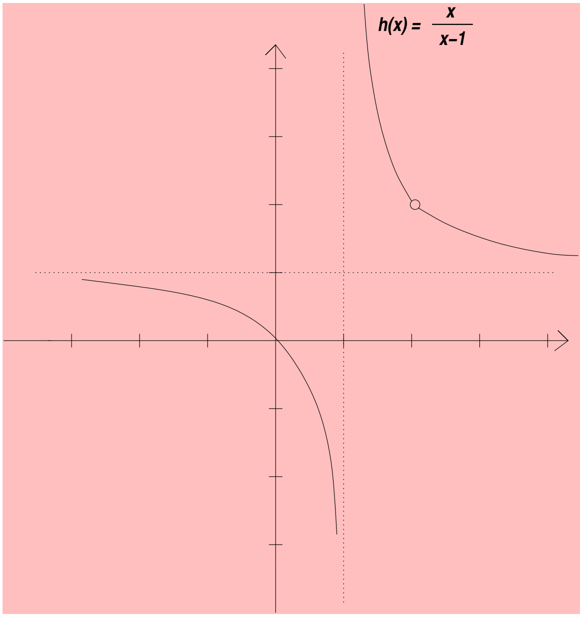


Figure 1.24 *The graph of* $f(x) = (x^2 + 2x)/(x^2 + x - 2)$.

Example 4.1 Sketch the graph of the symmetric function $f(x) = 1/(1+x^2)$.

Solution 4.1 We begin by considering the quadratic function obtained by taking the reciprocal function which is $g(x) = 1 + x^2$. The graph of this function has vertex $(0, 1)$. The function $g(x)$ is concave up and tends to infinity as x tends to positive and negative infinity. We conclude that $f(x)$ tends to zero from above as as x tends to positive infinity and it tends to zero from above when x tends to negative infinity. We graph both functions to observe the realtions of the reciprocals.

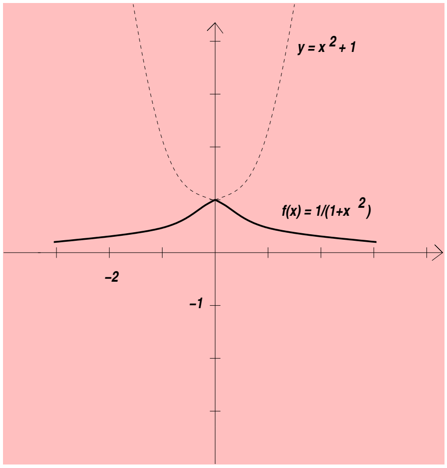


Figure 1.25 *The graph of* $f(x) = 1/(x^2 + 1)$.

Example 4.2 Sketch the graph of the symmetric function $f(x) = 1/(x^2 - 1)$.

Solution 4.2 Consider the quadratic reciprocal function which is $g(x) = x^2 - 1$. The graph of this function has vertex $(0, -1)$. The function $g(x)$ is concave up and has roots at $x = 1$ and $x = -1$. These roots will correspond to vertical assymptotes. As Also $g(x)$ tends to positive infinity as x tends to positive and negative infinity. We conclude that $f(x)$ tends to zero from above as x tends to positive and negative infinity. We have

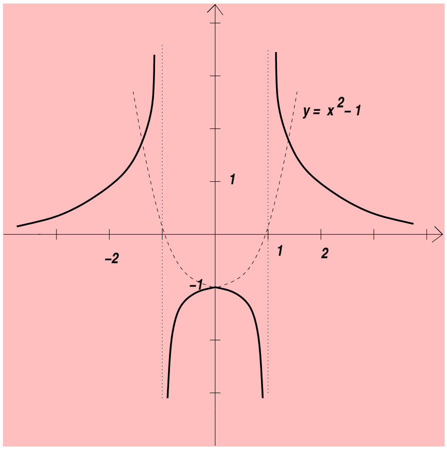


Figure 1.26 *The graph of* $f(x) = 1/(x^2 - 1)$.

Exercize 1.4

A. Sketch the graph of these simple functions reciprocal linear functions.

1. $f(x) = 1/(x-2)$.
2. $f(x) = 1/(x+1)$.
3. $f(x) = 1/(x+3)$.
4. $f(x) = 1/(x-3)$.

B. Sketch the graph of these rational functions and determine the center point.

1. $f(x) = x/(x-2)$.
2. $f(x) = x/(x+1)$.
3. $f(x) = x/(x+2)$.
4. $f(x) = (x+1)/(x+3)$.

C. Sketch the graph of these rational functions.

1. $f(x) = 1/(x^2+4)$.
2. $f(x) = 1/(x^2-4)$.

1.5 A Short Course in Trigonometric Functions

A ray is a directed extended line in the cartesian plane. The starting point of the ray is called the vertex. Two rays emanating from the same vertex determine an angle in the plane. If we designate one ray as the inital side an the other as the terminal side of the angle then the positive angle is measured in a counter-clockwise direction. A negative angle is measured clockwise.

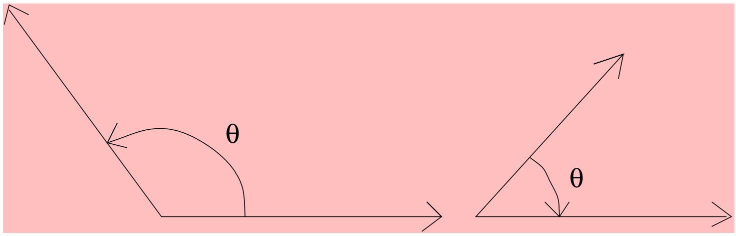


Figure 1.27 *A positive angle measure and a negative angle measure.*

In some cases the angle may be specified by one or more complete rotations between the initial ray and terminal angle.

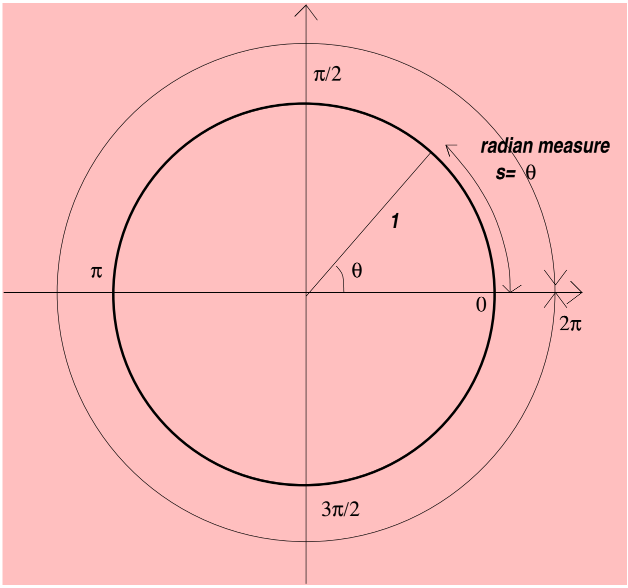


Figure 1.28 *A representation of radian measure.*

An angle is usually measured in degrees or radians. If the angle vertex is positioned at the center of a unit radius circle then the radian measure of the angle is the corresponding arclength on the arc of the circle. A straight line angle is 180^o. The corresponding arclength is π. This means we can convert from degrees to radian measure using the formula

$$\phi = \pi \frac{\theta}{180^o}, \quad \text{or,} \quad \theta = 180\Big(\frac{\phi}{\pi}\Big)^o.$$

Let (x, y) denote a point on the unit circle defined by

$$x^2 + y^2 = 1.$$

We wish to consider the angle θ that the line segment from $(0, 0)$ to (x, y) makes with respect to the x-axis where positive angles are measured in th counter clockwise direction.

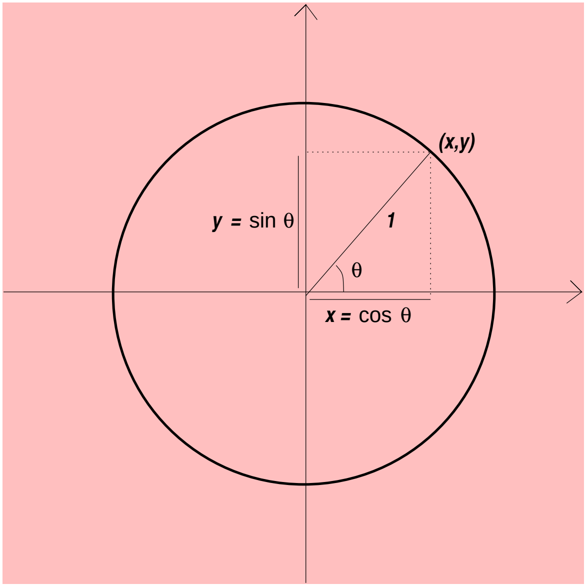


Figure 1.29 *A representation of the* $\sin\theta$ *and* $\cos\theta$*.*

We will assume that θ is a real number which is given by the arc length measured in radians along the unit circle associated to the angle. For instance if the angle is 90^o then the arc length along the circle is $\pi/2$ and if the angle is 180^o then the arc length along the circle is π. We will allow any angle satisfying $-\infty \leq \theta \leq \infty$ and associate the angle to the corresponding angle $\hat{\theta}$ satisfying $0 \leq \hat{\theta} < 2\pi$. We define

$$\cos(\theta) = \cos(\hat{\theta}) = x,$$

for the point (x, y) on the circle associated to the angle $\hat{\theta}$ and

$$\sin(\theta) = \sin(\hat{\theta}) = y.$$

We observe that this is related to the usual definition of the sine and cosine for an arbitrary right triangle. We then can define the function rules

$$\boxed{f(\theta) = \sin(\theta), \quad \text{or,} \quad g(\theta) = \cos(\theta),}$$

and observe that these rules give a well defined real valued function on the domain **R**. A graphical representation of the function can be given by using the variable form x in place of the variable form θ:

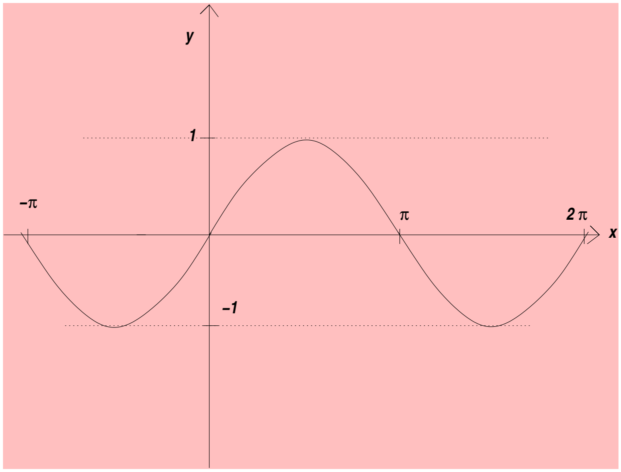


Figure 1.30 *A graphical representation of* $y = \sin(x)$.

The sine function can be used to represent a pure tone in the case of a sound wave.

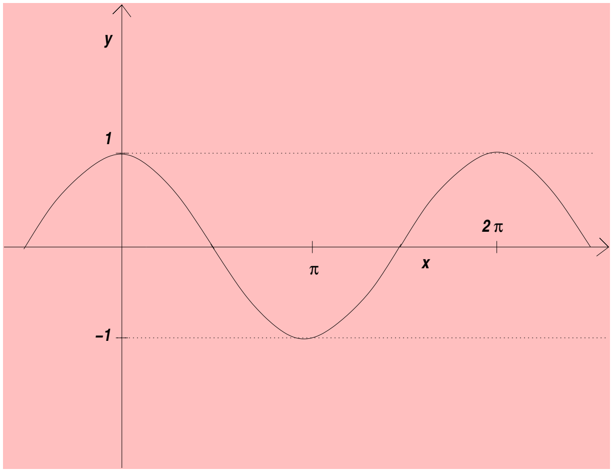


Figure 1.31 *A graphical representation of* $y = \cos(x)$

Defintion.

1. A function $f(x)$ is periodic if $f(x+L) = f(x)$ for some fixed value L and for all $x \in \mathbf{R}$.
2. The period T of a periodic function $f(x)$ is the smallest positive value such that $f(x+T) = f(x)$ for all $x \in \mathbf{R}$.
3. The frequency ω_o of a period function is
$$\omega_o = \frac{2\pi}{T}.$$
4. The amplitude A of a periodic function is
$$A = \mathbf{max}\{|f(x)|\}_{x\in\mathbf{R}}.$$
5. If $f(x) = g(x+\phi_o)$ for each $x \in \mathbf{R}$, then ϕ_o is called the *phase angle difference* if ϕ_o is the smallest positive value for which the relation holds.

The cosine function has the same shape as the sine function, but it is said to be out of phase from the sine function by an angle of $\pi/2$ radians.n This phase angle difference is represented by the equation

$$\cos(x - \frac{\pi}{2}) = \sin(x), \text{ or, } \sin(x + \frac{\pi}{2}) = \cos(x).$$

Note that the graphs of the functions are translates of each other such that the sine function graph is obtained by sliding the cosine graph forward a distance $\pi/2$.

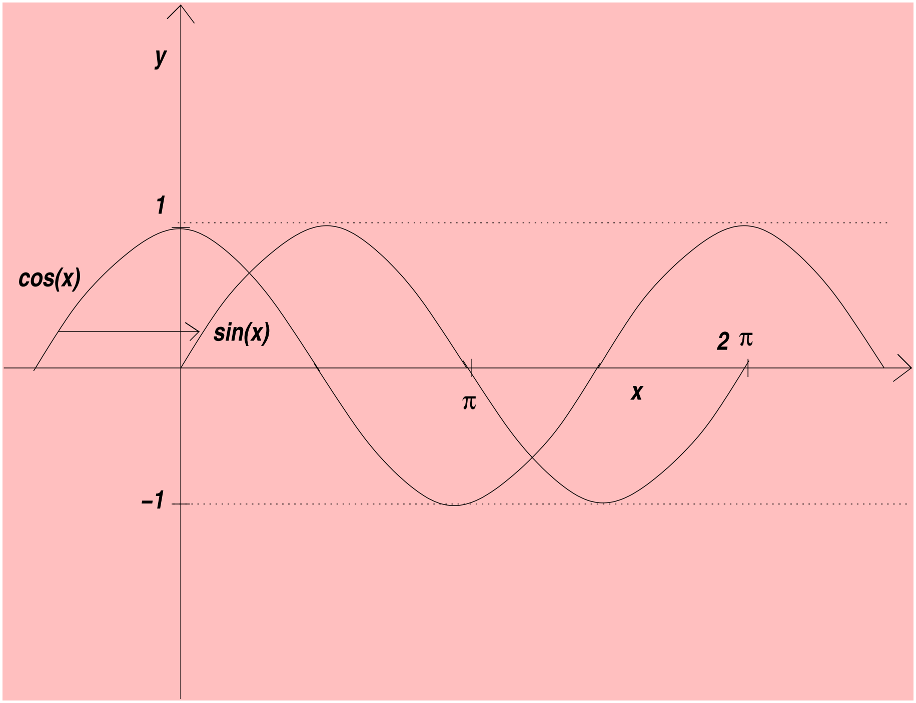


Figure 1.32 *A graphical representation of the phase relation.*

Joseph Fourier studied the conduction of heat in various materials. He found that the problem of heat transferal was related to the initial distribution of heat in the material written in the form of a summation of trigonometric functions called a trigonometric series. These functions were originally studied by Euler and Lagrange, but due to Fourier's monumental work they are now called Fourier series.

Any continuous function on a closed interval can be approximated by such a Fourier series. For this reason, bird and whale calls can be approximated, studied, and digitally recorded using a discrete Fourier series. In fact, music and human speech and any sort of sound wave can be digitized using the techniques of approximation by a finite sum of triginimetric functions.

The sound waves we hear are composed of air molecules at various pressure and density levels. These pressure levels determine how the ear drum vibrates. The vibration of the ear drum determines the sound one hears. It is a fact that both sound and visual perception are essentially passed through the nervous system as discrete data rather than continuous data. In particular, human hearing is encoded for the brain using what is equivalent to a finite number of sine-wave or pure tone function representations.

A pure pitch can be reprented by a simple sine-wave function such that the function is given with time, t, as the independent variable.

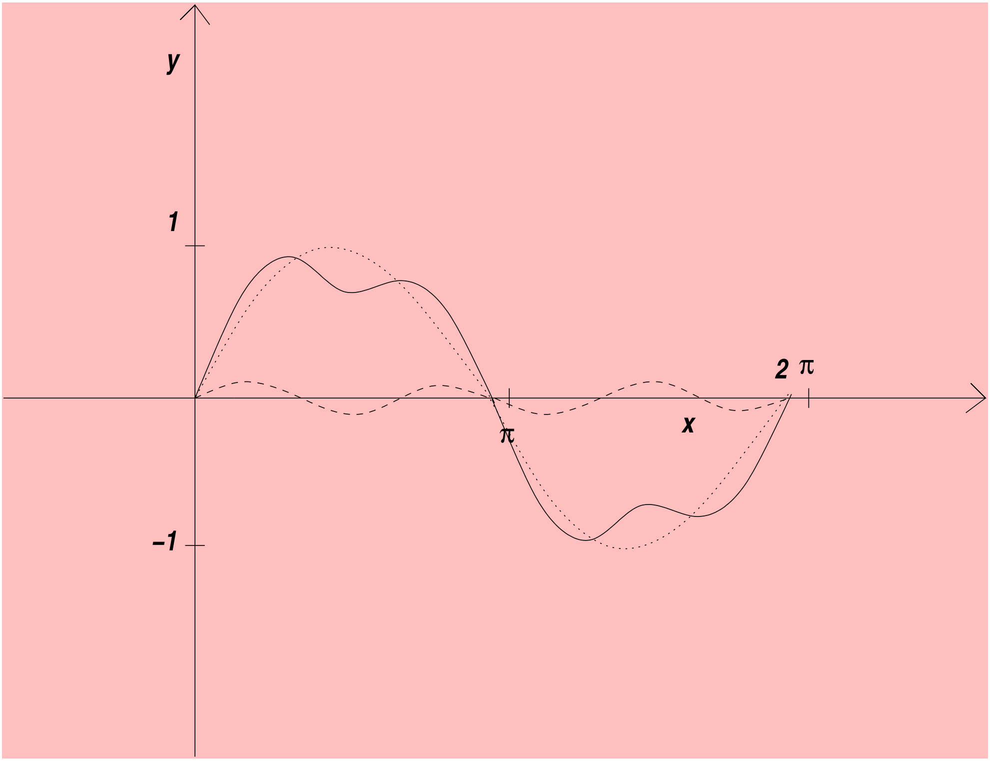


Figure 1.33 *A sound wave as a sum of two sine-waves.*

On the other hand, the same wave function can be represented by a histogram which gives the amplitude and the frequency where the frequency is now the independent function

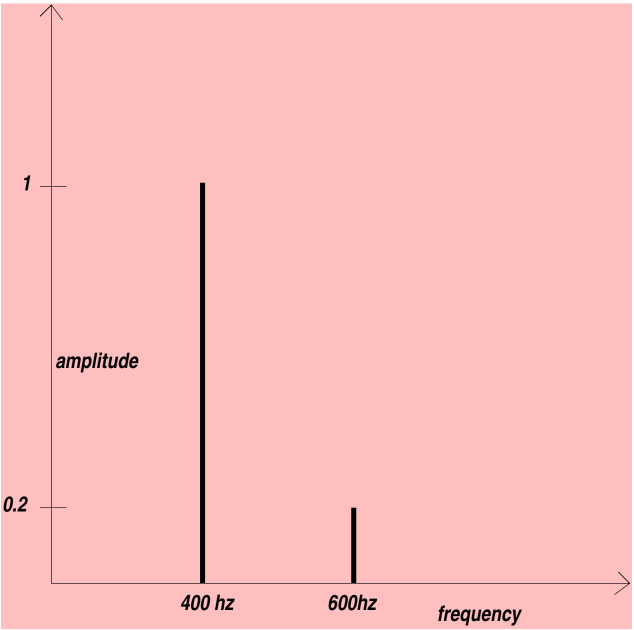


Figure 1.34 *The spectrograph form of the frequency and amplitude distribution.*

In this case the spectrograph has only two bars because the sound wave function is a sum of two simple sine-waves. Thus the function form is given by

$$f(t) = \sin(800\pi t) + \frac{\sin(1200\pi t)}{5},$$

where we have used that the frequency of the first term is 400Hz and the frequency of the second term is 600 Hz. More complicated functions will require more complicated spectrographs.

Example 5.1 Give the spectrograph of the function composed of trigonometric sum

$$f(t) = 2\sin(300\pi t) + \frac{\sin(400\pi t)}{2}$$

Solution 5.1 The formula for the frequency of a function of the form $\sin(\alpha t)$ is given by

$$\nu = \frac{\alpha}{2\pi}.$$

Thus we have,

$$\nu_1 = \frac{300\pi}{2\pi}\ \text{Hz} \ = 150\ \text{Hz}, \quad \text{and} \quad \nu_2 = \frac{400\pi}{2\pi}\ \text{Hz} \ = 200\ \text{Hz}.$$

The frequencies are 150 Hz and 200 Hz respectively. The amplitudes are $A_1 = 2$ and $A_2 = 0.5$ respectively. Therefore, the spectrograph is given by

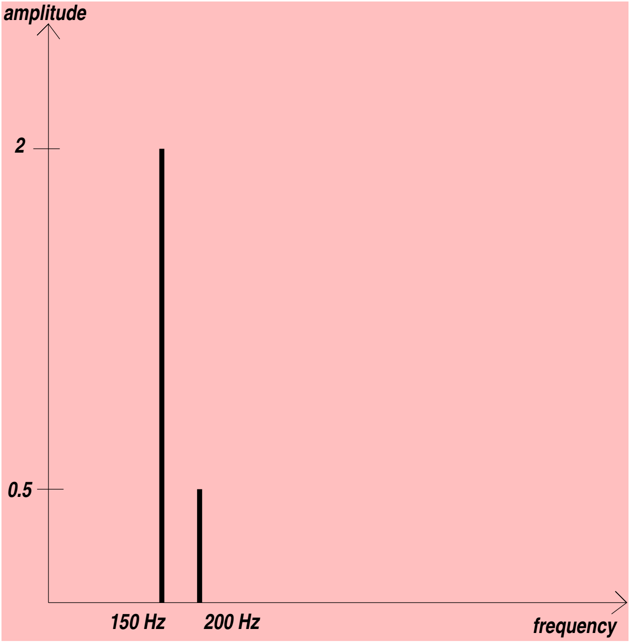


Figure 1.35 *The spectrograph of* $f(t) = 2\sin(300\pi t) + 0.5\sin(400\pi t)$.

Example 5.2 Given the spectrograph below find the trigonometric sum associated to it.

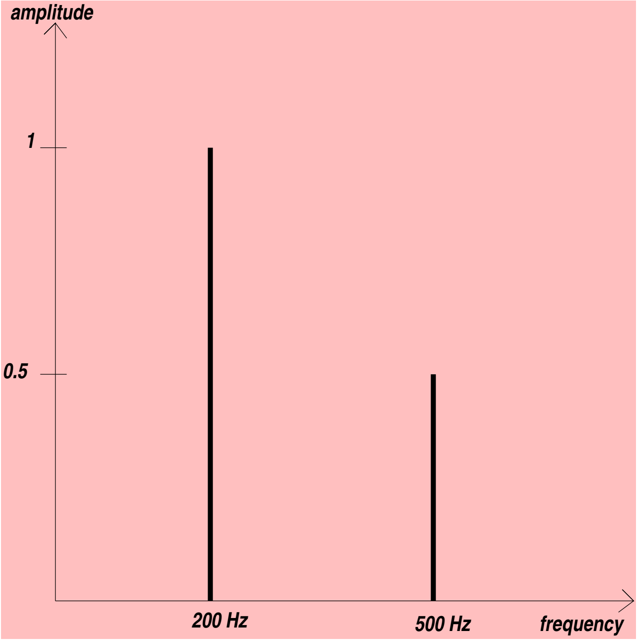


Figure 1.36 *The spectrograph of* $f(t) = 2\sin(300\pi t) + 0.5\sin(400\pi t)$.

Solution 5.2 The frequencies are 200 Hz and 500 Hz respectively. The amplitudes are 1 and 0.5. Now we can write

$$f(t) = \sin(400\pi t) + 0.5\sin(1000\pi t).$$

Thus far we have only considered the two elementary trigonometric functions $\sin(x)$ and $\cos(x)$. It is convenient to define other relation involving these functions the standard functions obtained from these elementary functions are the tangent function written $\tan(x)$; the cotangent function written $\cot(x)$; the secant function written $\sec(x)$; and the cosecant function written $\csc(x)$.

Definition.

1. The tangent function is defined by $\tan(x) = \frac{\sin(x)}{\cos(x)}$.
2. The cotangent function is defined by $\cot(x) = \frac{\cos(x)}{\sin(x)}$.
3. The secant function is defined by $\sec(x) = \frac{1}{\cos(x)}$.
4. The cosecant function is defined by $\csc(x) = \frac{1}{\sin(x)}$.

The period of $\tan(x)$ is π even though the period of $\sin(x)$ is 2π and the period of $\cos(x)$ is 2π.

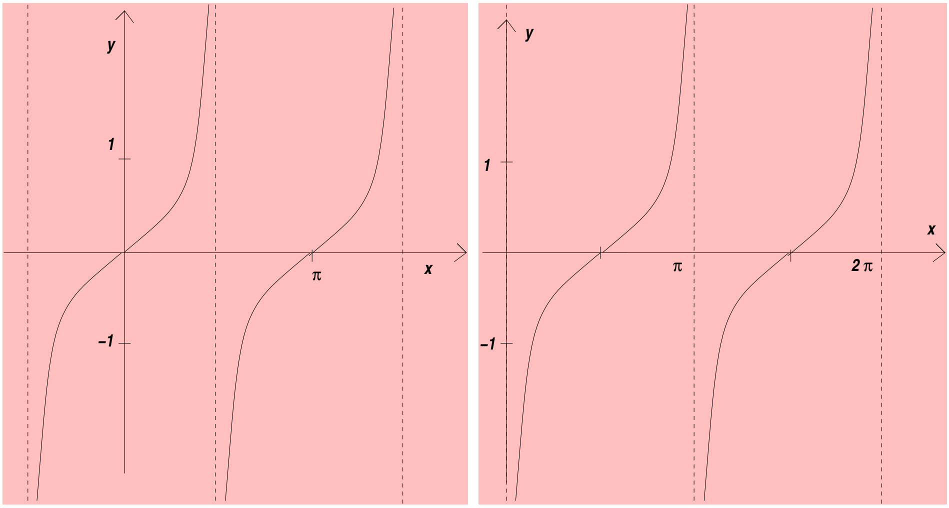


Figure 1.37 *The graph of* $f(x) = \tan(x)$ *and* $\cot(x)$.

The graph of $\cot(x)$ is given by the graph of the tangent function shifted to the right by $\pi/2$ units. Observe that the tangent function is undefined for the set

$$\{x \in \mathbf{R} \mid x = n\pi + \frac{\pi}{2},\ n \in \mathbf{Z}\} = \{\ldots,\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2}, \ldots\}.$$

The graph of the tangent tends to infinity as x tends to $n\pi + \pi/2$ from the left hand side as shown, for each $n \in \mathbf{Z}$. Also the graph of the tangent tends to negative infinity as x tends to $n\pi + \pi/2$ from the right hand side as shown, for each $n \in \mathbf{Z}$.

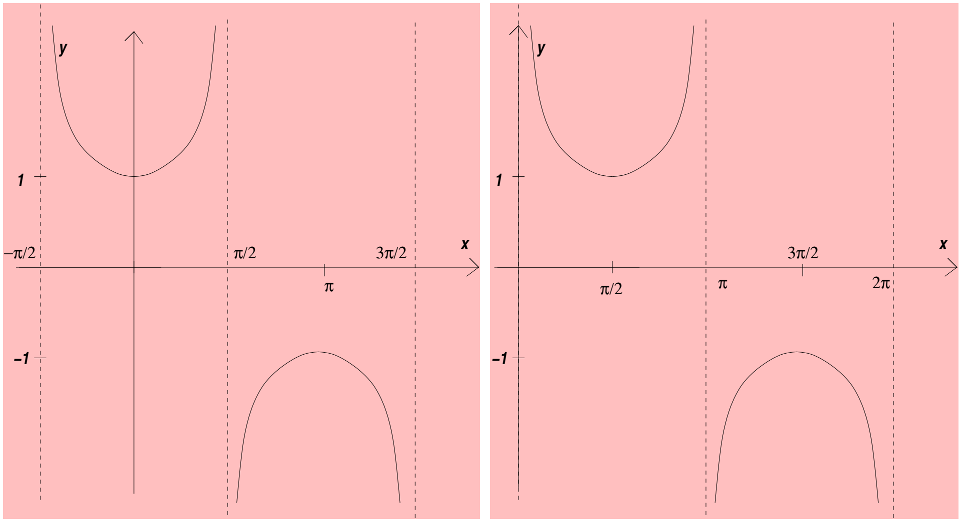


Figure 1.38 *The graph of* $\sec(x)$ *and the graph of* $\csc(x)$*.*

Trigonometry and similar triangles form a very useful field of study and are important tools for understanding the environment.

> ***In the 18th century Sir George Evert and Mr. Lambton undertook the arduous task of surveying the unknown regions of India. The survey was conducted by constructing triangles with specific points on the topography . Using field equipment the distances of the sides of each triangle and the height above sea level of each point was calculated The survey team was able to calculate the height of Mount Everest to be 29000 feet. This calculation is within 30 feet of the modern calculation using satellite technology***

Consider the following example.

Example 5.3 A group of hikers wish to climb a peak but are not sure of its height above the plain. One of the hikers is able to estimate the height of some nearby trees and determines that there are two tree of height 100 feet that are 1000 feet apart in a straight line with respect to the peak. The hiker lines the top of the closer tree with the peak and measures the distance to the tree as 100 feet. The hiker lines the top of the second tree up with the peak and measures the distance to the tree from that spot as 173 feet. Find the height of the peak.

Solution 5.3 We observe that the first angle is 45^o and the second angle is about 30^o. Assume that D_1 is the distance from the first spot to the point below the peak. Let D_2 be the distance from the second spot to the point below the peak. Let H denote the height of the peak. We have the folowing relations

$$\frac{H}{D_1} = 1, \quad \frac{H}{D_2} = \frac{100}{173}, \quad D_1 + 1000 = D_2$$

and

$$173\frac{H}{D_1 + 1000} = 100, \quad \text{or}, \quad 173H = 100(H + 1000).$$

This gives

$$H = \frac{100000}{73} \approx 1369 \text{ ft.}$$

Exercize 1.5

A. Sketch the graph of these trigonometric functions.

1. $f(x) = 1 + \cos(x)$.
2. $f(x) = 2\cos(x)$.
3. $f(x) = \sin(x - \pi)$.
4. $f(x) = 3\sin(x)$.
5. $f(x) = 2\sec(x)$.
6. $f(x) = 4\csc(x)$.

B. Graph the spectrograph for these sine functions.

1. $f(x) = 2\sin(100\pi t) + \sin(440\pi t)$.
2. $f(x) = 3\sin(500\pi t) + 4\sin(840\pi t)$.
3. $f(x) = \sin(1000\pi t) + 2\sin(400\pi t)$

C. Sketch the graph of these rational functions.

1. $f(x) = 1/(x^2 + 4)$.
2. $f(x) = 1/(x^2 - 4)$.

1.6 Inverse Functions

Definition. A function is said to be 1-1, or *one-to-one*, if for each y in the range the is exactly one x in the domain such that $y = f(x)$.

Inverse Functions A 1-1 function $f(x)$ on a domain D_f with range D_g is said to have an inverse function if there exists a function $g(x)$ defined on D_g with range D_f such that

$$f \circ g(x) = x \text{ for } x \in D_g \text{ and } g \circ f(x) = x \text{ for } x \in D_f.$$

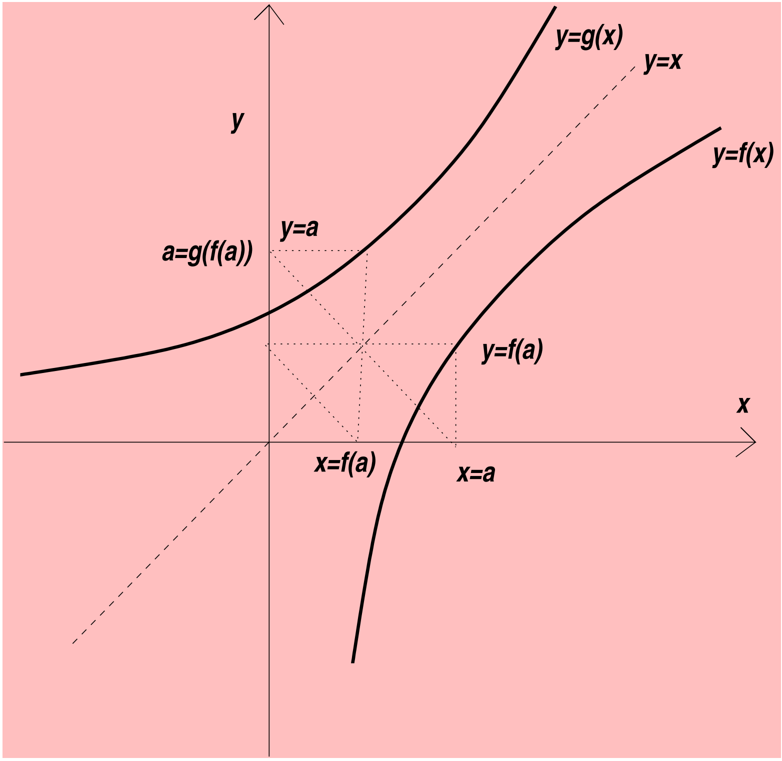


Figure 1.39 *A graphical representation of the inverse function relation*

Definition. A pair of functions (possibly the same function applied twice) $f(x)$ and $g(x)$ are inverses of each other if the functions are 1-1 on their respective domains and the composition functions $f \circ g(x)$ and $g \circ f(x)$ are defined such that

$$f \circ g(x) = x, \quad \text{and} \quad g \circ f(x) = x.$$

Example 6.1 Find the inverse function for $f(x) = \sqrt{x}$. Graph the function and its inverse and determine the domain of the inverse function.

Solution 6.2 We observe that the domain of $f(x)$ is $D_f = \{x \geq 0\}$. We set $y = \sqrt{x}$ and solve for x as a function of y. Then

$$y^2 = x.$$

This implies that $g(y) = y^2$ defines the rule for the function. But the argument used to define the rule is not important so we write the function as a rule in terms of the variable x. Then $g(x) = x^2$ with $D_g = \{x \geq 0\}$. We obtain the domain for $g(x)$ by observing that the function must be 1-1 and the domain $\{x \leq 0\}$ will not work. Also if we apply our knowledge of the graphical representation it is clear that we have chosen the correct domain.

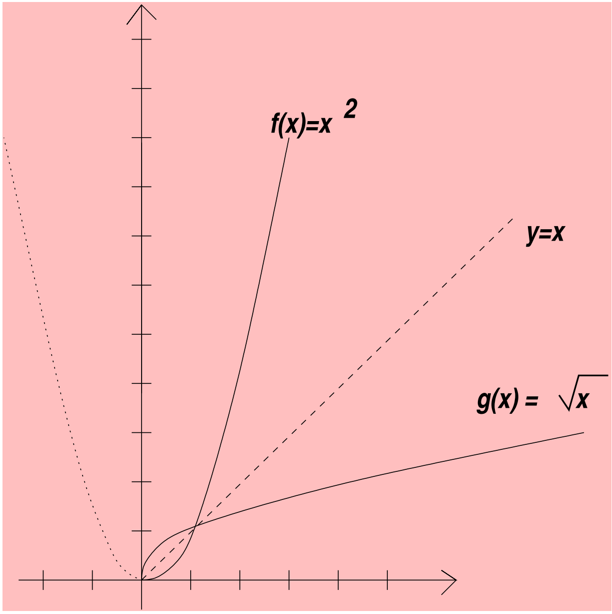


Figure 1.40 *A graphical representation of the inverse function relation*

Now we check the composition to see that it satisfies the definition. Applying the rule method we have

$$f \circ g(x) = f(g(x) = f(x^2) = \sqrt{x^2} = |x| = x,$$

since $x \geq 0$. Note that the last step in the computation above is only possible when $x \geq 0$. We must also check the other composition

$$g \circ f(x) = g(f(x) = g(\sqrt{x}) = (\sqrt{x})^2 = x.$$

Example 6.3 Find the inverse function for $f(x) = \sqrt{1 - x^2}$. Describe the function and graph it along with its inverse and determine the domain of the inverse function.

Solution 6.3 The function is the upper semi-circle of radius 1. Thus we can see that the domain of $f(x)$ is $\{-1 \leq x \leq 1\}$. But this does not correspond to a 1-1 function. We restrict the domain to $D_f = \{0 \leq x \leq 1\}$. We set

$$y = \sqrt{1 - x^2}$$

and solve for x as a function of y. Then

$$y^2 = 1 - x^2, \quad \text{and} \quad x^2 = 1 - y^2,$$

which implies that $g(y) = \sqrt{1 - y^2}$ defines the rule for the function. Applying the rule method we have $g(x) = \sqrt{1 - x^2}$ and consequently, $f(x) = g(x)$ with $D_f = D_g$.

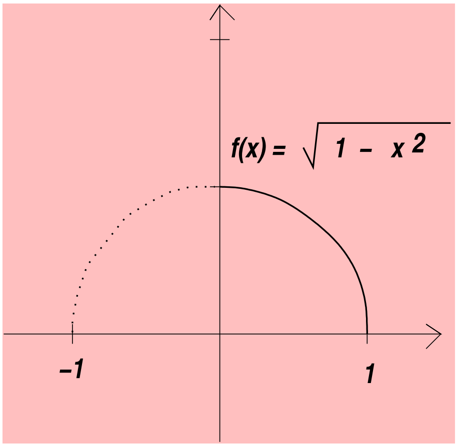


Figure 1.41 *A graphical representation of the inverse function relation*

We consider the composition of the functions to verify that it satisfies the definition. Applying the rule method we have

$$f \circ g(x) = f(g(x) = f(\sqrt{1-x^2}) = \sqrt{1-(1-x^2)} = |x| = x,$$

since $x \geq 0$. We must also check the other direction, but this is clear since $f(x)$ is its own inverse.

Exercises 1.6

A. Determine a domain of the function that is 1-1.

1. $f(x) = x^2 + 1$
2. $f(x) = \cos(x)$
3. $f(x) = \sin(x)$
4. $f(x) = x^3$
5. $f(x) = x^3 + 1$
6. $f(x) = \sqrt{x+1}$

B. Determine an inverse function on the given domain.

1. $f(x) = x^2 + 1,\ x \geq 0.$
2. $f(x) = x + 1\ x \in \mathbf{R}.$
3. $f(x) = \sqrt{x+2},\ x \geq -2.$
4. $f(x) = x^3,\ x \in \mathbf{R}.$
5. $f(x) = x^3 - 1,\ x \in \mathbf{R}.$
6. $f(x) = \sqrt{x^3}, x \geq 0.$

C. Show that the functions are inverses.

1. $f(x) = (x+1)^3,\ g(x) = (x-1)^{1/3},\quad D_f = \{x \in \mathbf{R}\}.$
2. $f(x) = x - 1,\ g(x) = x + 1,\quad D_f = \{x \in \mathbf{R}\}.$
3. $f(x) = \sqrt{x-1}, g(x) = x^2 + 1,\ D_f = [1, \infty).$

1.7 Graphing Techniques

Graphical Symmetry

1. A function $f(x)$ is said to be an *even* function if $f(-x) = f(x)$.
2. A function $f(x)$ is said to be an *odd* function if $f(-x) = -f(x)$.

Observe that any quadratic function of the form $f(x) = ax^2 + c$ is an even function. The line of symmetry is the y-axis and the vertex lies on the y-axis. In fact any polynomial with strictly even exponents is an even function. Any cosine function of the form $f(x) = \cos(\alpha x)$ is an even function.

An polynomial with strictly odd exponents is an odd function and is symmetric with respect to the origin. For example, a polynomial of the form $f(x) = ax^3 - bx$ is an odd function.

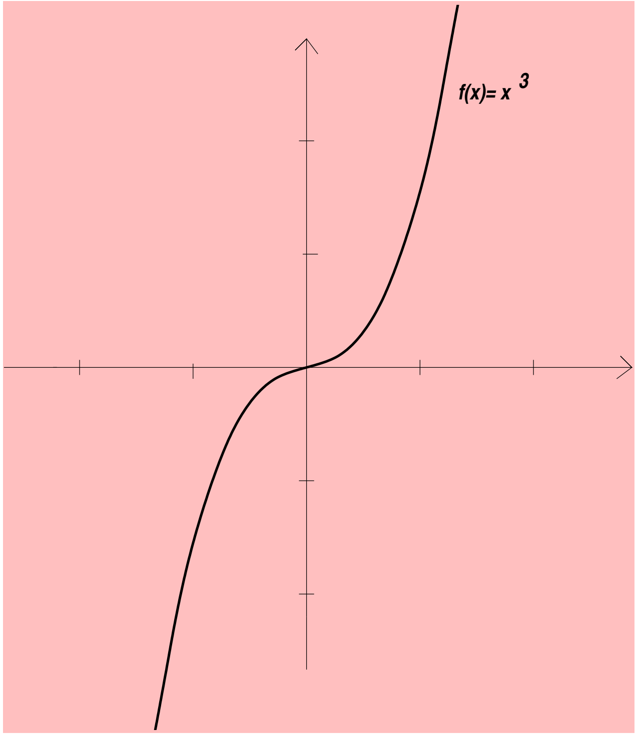


Figure 1.42 *Graphing the odd function* $f(x) = x^3$.

Graphical Translations

1. The graph of the function

$$y = f(x - a)$$

can be obtained by sliding the graph $y = f(x)$ by a distance a to the right if $a > 0$ or by sliding the graph of $y = f(x)$ by a distance a to the left if $a < 0$.

2. The graph of the function

$$y - b = f(x)$$

can be obtained by sliding the graph $y = f(x)$ by a distance b upward if $b > 0$ or by sliding the graph of $y = f(x)$ by a distance b downward if $b < 0$.

We will justify this fact by considering the case

$$y - b = f(x - a), \quad a, b > 0.$$

We make the varibale substitutions $y = v + b$ and $x = u + a$ then we obtain the origin $(0, 0)^*$ of the coordinate system for the plane with coordinates (u, v) is at the point (a, b) in terms of the usual coordinate system we have $v = f(v)$, and

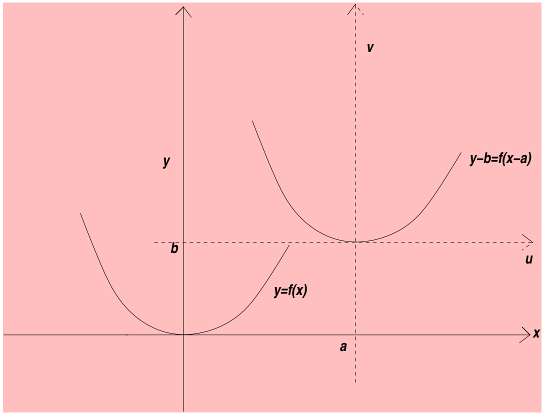


Figure 1.43 *Graphing* $y - b = f(x - a)$ *as a translate of* $y = f(x)$.

As an example of this method let us return to the study of ration functions. Recall that a ration function is a quotient of polynomials such that the domain is the set of points for which the denominator is non-zero. We have already graphed some simple cases of rational functions. We consider here the more general case and we outline the basic strategy for graphing.

Proceedure for Graphing Rational Functions

1. Find the domain by removing the roots of the denominator polynomial from **R**.

2. Simplify the rational function by cancelling like factors from the numerator and denominator.

3. For each root in the simplified denominator polynomial there is a vertical assymptote for the function which can be marked on the graph by a vertical dotted line.

4. Note the roots of the reduced numerator polynomial correspond to roots of the rational function.

5. Divide the real line into the intervals separated by roots of the numerator ploynomial and the denominator polynomial.

6. Construct a table using these intervals to determine the sign of the function in each interval.

7. Use the table to determine whether the function tends to infinity or negative infinity at the assymptotes.

8. Compute the values of the function at any points that are easily computed.

Now consider the case $h(x) = x/(x-1)$. This function can be written in the form $h(x) = (x-1+1)/(x-1) = 1 + 1/(x-1)$. We can obtain this graph by translating the graph of $g(x)$ by one unit upward as below.

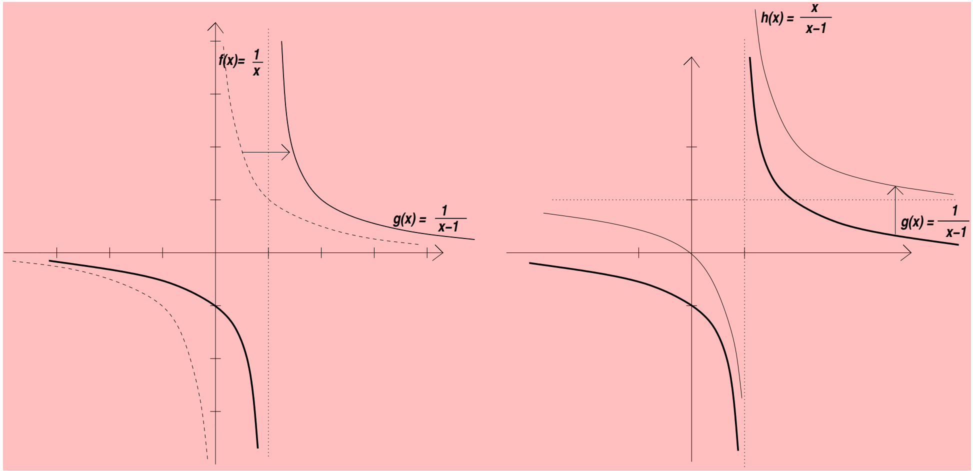


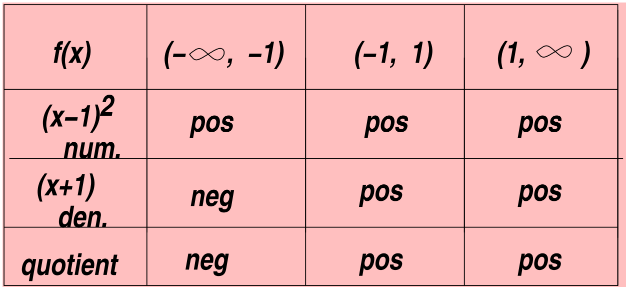
Figure 1.44 *The graph of* $f(x) = \frac{x}{(x-1)}$.

Example 7.1 Find the domain and graph the rational function

$$f(x) = \frac{(x-1)^2(x+2)}{x^2-1}.$$

Solution 7.1 The roots of the denominator polynomial are at $x = 1$ and $x = -1$ and so the domain is $\mathbf{R} - \{-\mathbf{1}, \mathbf{1}\}$. There is only one root of the numerator and it is at $x = 1$. The rational function on its domain simplifies to

$$f(x) = \frac{(x-1)^2}{x+1}, \quad x \neq 1.$$



f(x)	(−∞, −1)	(−1, 1)	(1, ∞)
$(x-1)^2$ num.	pos	pos	pos
(x+1) den.	neg	pos	pos
quotient	neg	pos	pos

Table 1.2

The numerator root at $x = 1$ is of degree 2 and is not in the domain. This is an even root and so the function will not change sign across this root. To graph this function it is convenient to graph the numerator polynomial and the denominator polynomial for the reduced form. We will do this using a differnt stylse to distinguish it from the graph of the function. The denominator root at $x = -1$ is a root of odd degree and the function will change sign across this root and tend to an infinity on either side.

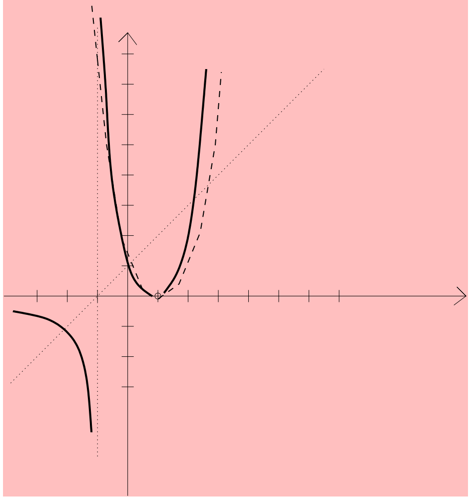


Figure 1.45 *The graph of* $f(x) = (x-1)^3/(x^2-1)$.

Example 7.2 Find the domain and graph the rational function

$$f(x) = \frac{x^2 - 2x - 3}{x^2 - 3x + 2}.$$

Solution 7.2 The denominator polynomial can be written as $(x-2)(x-1)$. The domain is $\mathbf{R} - \{\mathbf{1}, \mathbf{2}\}$ since $x = 1$ and $x = 2$ represent the roots of the denominator. The numerator can be written as $(x+1)(x-3)$. In this case there are no common roots and so the function does not simplify by cancellation. We write the function in factored form:

$$f(x) = \frac{(x-3)(x+1)}{(x-2)(x-1)}.$$

To graph this we wish to determine the sign of the function in the intervals separated by the roots of the function.

f(x)	(−∞, −1)	(−1, 1)	(1, 2)	(2, 3)	(3, ∞)
(x−3)(x+1) num ·	pos	neg	neg	neg	pos
(x−2)(x−1) den.	pos	pos	neg	pos	pos
quotient	pos	neg	pos	neg	pos

Table 1.3

Using the table and noting the roots in the denominator, we can sketch the graph of the function as follows

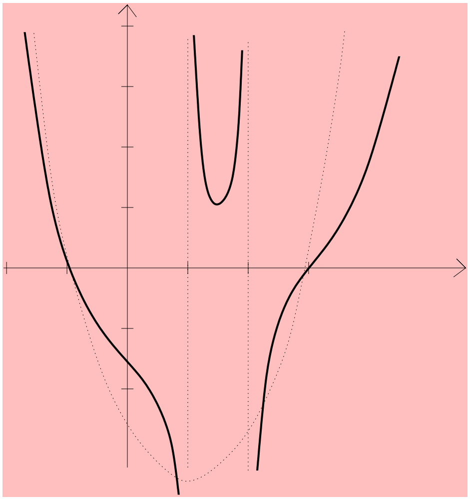


Figure 1.46 *The graph of* $f(x) = (x-3)(x+1)/(x-2)(x-1)$.

Below we sketch the reasoning for constructing the graph above.

1. Note that the function is positive for $x < -1$ as we have indicated.
2. The function is zero at $x = -1$ since this is a root of the numerator.
3. The function tends to minus infinity as x tends to 1 on the left hand side since $x = 1$ is a root of the denominator and the function is negative.

4. The function is positive between $x = 1$ and $x = 2$ and tends to positive infinity at each end of the interval $(1, 2)$ since these are the roots of the denominator.

5. The function tends to minus infinity as x tends to 2 from the right hand side since the function is negative and $x = 2$ is a root of the denominator.

6. The function increases to the value zero at $x = 3$ and continues to increase.

In some cases it is useful to apply long division on polynomials to simplify the function before graphing.

Example 7.3 Use long division to simplify the rational function and use this to graph the function.

Solution 7.3 We apply long division to the rational function to obtain

$$\begin{array}{rl} & \quad 1 \;\; R \;\; 1 \\ \cline{2-2} x^2 - x - 2 \, \big) & x^2 - x - 1 \\ & -(x^2 - x - 2) \\ \cline{2-2} & \qquad\quad 1 \end{array}$$

Now we can write the long division result using the quotient remainder formula:

$$f(x) = 1 + \frac{1}{(x-2)(x+1)}.$$

The analysis of roots for $g(x) = 1/(x^2 - x - 2)$ is

1. The roots of the denominator are $x = -1$ and $x = 2$.

2. The function $g(x)$ is positive for $x < -1$, negative when $-1 < x < 2$ and positive when $x > 2$.

3. The function $g(x)$ tends to positive infinity as x tends to -1 from the left hand side.

4. The function $g(x)$ tends to negative infinity on each side of the interval $(-1, 2)$
5. The function $g(x)$ tends to positive infinity as x tends to 2 from the right hand side.

Now we have the graph of $g(x)$ based on this analysis which we translate upward by one unit obtaining the graph of $f(x)$.

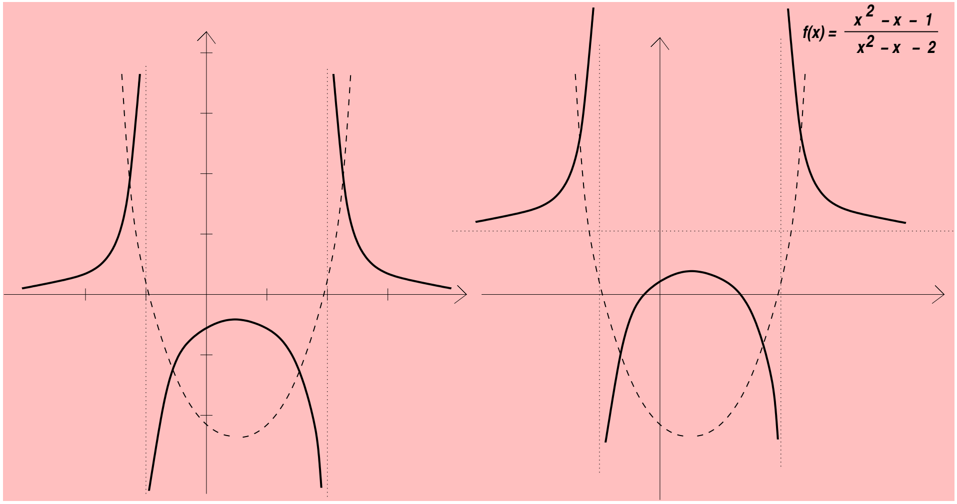


Figure 1.47 *The graph of* $f(x) = 1 + 1/(x^2 - x - 2)$.

CHAPTER 2.
LIMITS AND CONTINUITY

2.1 Real Valued Sequences and Their Limits

A *sequence* is a real valued function with a restricted domain set, namely $\mathbf{N}$. We write a sequence in the form

$$\boxed{f : \mathbf{N} \to \mathbf{R}, \text{ or } f_n \in \mathbf{R}, \ n \in \mathbf{N}}$$

We often write $\{f_n\}_{n=1}^{\infty}$ to signify the sequence.

Now consider the following problem posed by Leonardo Fibonacci in the year 1202. A certain rabbit and its descendants, which do not die, produce one rabbit each month. A rabbit begins to produce in the second month after birth. How many rabbits have been produced as a function of the number of months.

We observe that there is 1 rabbit the first month, the same rabbit the second month and 2 rabbits the third month. In the fourth month the first rabbit produces another rabbit but the second rabbit does not. Consequently, there are 3 rabbits after four months. Now there are 5 rabbits in the fifth month because the second rabbit begins to produce in that month.

It should now be clear that the number of rabbits produced each month is equal to the number of rabbits that had been produced at least one month earlier; these rabbits produce new rabbits in the given month; and the number of rabbits that had been produced by the previous month; these rabbits are the carry over population. We have the following recursion formula for the Fibonacci sequence p_n, where p_n represents the model population of rabbits in month n

$$\boxed{p_n = p_{n-1} + p_{n-2}, \quad n \geq 3.}$$

Unfortunately, this does not explicitly tell us the function of this sequence. It does allow us to evaluate the function for any n provided we have time to do the calculation. Observe that the sequence begins with

$$\{p_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610 \ldots\}.$$

In particular, $p_{15} = 610$.

A sequence is usually define by a rule such as

$$f_n = f(n), \text{ where } f : \mathbf{R} \to \mathbf{T}$$

is some real valued function.

Definition.

1. We define the *harmonic sequence* by

$$f_n = \frac{1}{n}, \quad n \in \mathbf{N}, \quad \text{or} \quad \{\frac{1}{n}\}_{n=1}^{\infty}.$$

2. We define the *natural number sequence* by

$$f_n = n \quad n \in \mathbf{N} \quad \text{or} \quad \{n\}_{n=1}^{\infty}.$$

3. We define an *arithmetic sequence* by

$$f_n = a + bn \quad n \in \mathbf{N}, \ a, b \in \mathbf{R}, \quad \text{or} \quad \{a + bn\}_{n=1}^{\infty}.$$

4. We define a *geometric sequence* by

$$f_n = r^n \quad n \in \mathbf{N}, \ r \in \mathbf{R}, \quad \text{or} \quad \{r^n\}_{n=1}^{\infty}.$$

Idea of a Sequence Limit A sequence $\{f_n\}_{n=1}^{\infty}$ is said to converge to a limit L if we can force f_n to be as close to L as we like provided that n is chosen sufficiently large.

Example 1.1 Show that the geometric sequence $\{1/2^n\}_{n=1}^{\infty}$ converges to zero.

Solution 1.1 First note that

$$\frac{1}{2^{n+1}} = \frac{1}{2^n} \cdot \frac{1}{2} < \frac{1}{2^n}.$$

Therefore, the distance of the sequence from zero decreases by one half at each step. Since 2^n is unbounded it follows that $1/2^n$ is as close to zero as we wish provided that we choose n large enough. To be more precise, let ϵ represent any positive number less than 1. Then $1/\epsilon$ is larger than 1, but it is less than 2^n for some choice of n. Thus $1/2^n < \epsilon$ for some choice of n.

Example 1.2 Show that any geometric sequence $\{r^n\}_{n=1}^{\infty}$ with $r < 1$ converges to zero and for $r = 1$ the geometric sequence converges to 1.

Solution 1.2 If $r = 1$ then $f_n = r^n = 1$ for all $n \in \mathbf{N}$. Now for $r < 1$, we have as before,

$$rn + 1 = r^n \cdot r < r^n \quad \text{for} \quad n \in \mathbf{N}.$$

Assume that $1/r = 1 + \mu$ such that $\mu > 0$. Applying the foil method repeatedly

$$(1+\mu)^2 = 1 + 2\mu + \mu^2, \quad (1+\mu)^3 = 1 + 3\mu + 3\mu^2 + \mu^3, \quad \ldots,$$

and truncating to the first two terms we have

$$\frac{1}{r^n} = (1+\mu)^n > 1 + n\mu.$$

This implies that $1/r^n$ is unbounded (i.e., is as large as we like for some choice of n). Thus r^n is as close to zero as we like for an appropriate choice of n.

Example 1.3 Show that the harmonic sequence converges to zero.

Solution 1.3 Since

$$\frac{1}{n+1} - 0 < \frac{1}{n},$$

we can see that $f_{n+k} = 1/(n+k)$ is as closer than the distance $1/n$ for each $k \geq 1$. Now observe that $1/(2n)$ is half the distance from zero as $1/n$. Thus it is clear from geometric considerations that we can choose $1/n$ to be as close to zero as we like.

Example 1.4 Assume that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L and the sequence $\{b_n\}_{n=1}^{\infty}$ satisfies $L < b_n < a_n$. Show that $\lim_{b \to \infty} = L$.

Solution 1.4 Since $\lim_{n\to\infty} a_n = L$, we can choose some large number N such that the distance $|L - a_n|$ is as small as we like for $n > N$. It follows that $|L - b_n|$ has an even smaller value for $n > N$ due to the hypothesis $L < b_n < a_n$. In other words,

$$0 < b_n - L < a_n - L, \quad \text{for} \ \ n > N.$$

Recall that a set of real numbers, A, is said to be *bounded* if there exists a positive real number K such that

$$|a| \leq K, \ \text{when} \ \ a \in A.$$

In this case K is an *upper bound* for the set A. This means that K is greater than or equal to each element $a \in A$. We also have that $-K$ is a *lower bound* for the set A. This means that $-K \leq 0$ is less than or equal to each element $a \in A$. We will assume that each bounded set of real numbers has a *least upper bound* and a *greatest lower bound*.

We now observe the following facts. The first three are a result of the existence of the least upper bound and the greatest lower bounds. The other results will be proven later in a more general context.

Remark 2.1. *Let* $\{f_n\}_{n=1}^{\infty}$ *and* $\{g_n\}_{n=1}^{\infty}$ *denote convergent sequences such that* $\lim_{n\to\infty} f_n = L$, *and* $\lim_{n\to\infty} g_n = M$.

1. *Every bounded increasing sequence converges to its least upper bound* L.
2. *Every bounded decreasing sequence converges to its greatest lower bound* L.
3. *If a sequence is unbounded it does not converge.*
4. $\lim_{n\to\infty}(f_n \pm g_n) = L \pm M$.
5. $\lim_{n\to\infty}(f_n \cdot g_n) = L \cdot M$.
6. $\lim_{n\to\infty}(f_n/g_n) = L/M$ *where* f_n/g_n *is defined for each* n *and* $M \neq 0$.

Example 1.5 Assume that a colony of bacteria in a Petri dish doubles in size every 24 hours. Assume that the size is initially 1 cm^2. Find a sequence that models the size of the colony as a function of the number of days.

Solution 1.5 Since the colony size is initially 1 cm^2 we have that $p_1 = 1$. Then $p_2 = 2$ and $p_3 = 4$. In general $p_n = 2^{n-1}$, where n represents the day of the measurement. The sequence is $\{2^{n-1}\}_{n=1}^{\infty}$. Of course, the size of the colony will not grow at this rate indefinitely and so the model is only good for a relatively small number of days.

Example 1.6 Assume that each ring of a tree trunk representing the growth of one season is 5 mm thick. Starting in year 5, the diameter of the tree is 3 cm. Find a sequence to represent the growth of the diameter of the tree and use this to estimate the diameter of the tree after a total of 100 years of growth.

Solution 1.6 The initial diameter is 3 cm and the diameter of the tree varies by 1 cm each year since the tree ring width changes by 0.5 cm per year. Thus the sequence is $\{3 + (n-5)\}_{n=5}^{\infty} = \{-2 + n\}_{n=5}^{\infty}$. This implies that the diameter of the tree after 100 years of growth is

$$D = -2 + 100 = 98\text{cm}.$$

Assume that two species interact as a population of predators, y, and a population of prey, x. Suppose that it is know that they satisfy the following recursion relation

$$\boxed{\begin{aligned} x_{n+1} &= [[(1 - \frac{y_n}{\alpha})\frac{x_n}{\mu}]] + x_n, \\ y_{n+1} &= [[(\frac{x_n}{\beta} - 1)\frac{y_n}{\mu}]] + y_n, \quad \alpha, \beta, \mu > 0. \end{aligned}}$$

This relation is a relatively simple form of the predator-prey relations called the Lotka-Volterra equations. Observe that there are many possible solutions for this recursion relation. In other words, there are many possible set of sequences $\{x_n\}$, $\{y_n\}$ and each set of sequences depends on the initial conditions. For instance, if $x_1 = \beta$ and $y_1 = \alpha$ then

$$x_{n+1} = x_n = \beta, \quad \text{and} \quad y_{n+1} = y_n = \alpha, \quad n \in \mathbf{N}.$$

This is a static or equilibrium solution. This means that the solution is constant for all time. On the other hand if $x_1 = \beta$ and $y_1 < \alpha$ then the number of prey increase while the number of predators remain constant. The figure below gives an account of the dynamics of the problem.

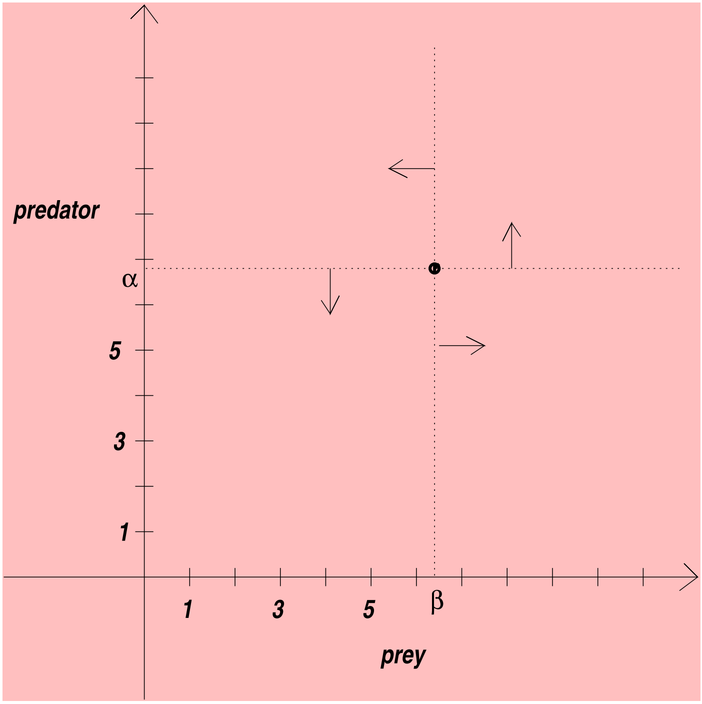


Figure 2.1 *The dynamic solution and the static solution at* (β, α)

This system models the predator-prey relation under limited circumstances. We will consider a real valued function model when we have developed the theory of differentiation.

Example 1.7 Assume that $x_1 = 10000$ and $y_1 = 50$ represent the initial values for a simple predator-prey realtion with recursion relations

$$x_{n+1} = [[(1 - \frac{y_n}{100})x_n/20 + x_n]], \quad y_{n+1} = [[(\frac{x_n}{100^2} - 1)y_n/20 + y_n]],$$

where $[[a]]$ is the largest integer less than the given real number. Calculate the first three sets of terms of the sequence.

Solution 1.7 The first set is given. Namely $x_1 = 10000$ and $y_1 = 50$. Using the recursion formula above , we have

$$\begin{aligned}
x_2 &= [[(1 - \frac{50}{100})\frac{1000}{2}]] + 10000, = 10000 + [[\frac{1000}{4}]] = 10250,\\
y_2 &= [[(\frac{10000}{100^2} - 1)5/2]] + 50 = 50,\\
x_3 &= [[(1 - \frac{50}{100})\frac{1025}{2}]] + 10250, = 10250 + [[\frac{1025}{4}]] = 10506,\\
y_3 &= [[(\frac{10250}{100^2} - 1)5/2]] + 50 = 50.
\end{aligned}$$

This completes the computation.

If one were to compute the first 150 terms of the predator-prey sequence from the above example, one would see the maximum prey would be about 20412 when $n = 37$, whereas the predator population is about 103. The predator population hits a maximum of 202 when $n = 59$ and the prey population is about 10548. The prey has a minimum population of about 3494 when $n = 95$ and the prey population is 98. When $n = 150$ the predator and prey populations return to values very close to the initial values.

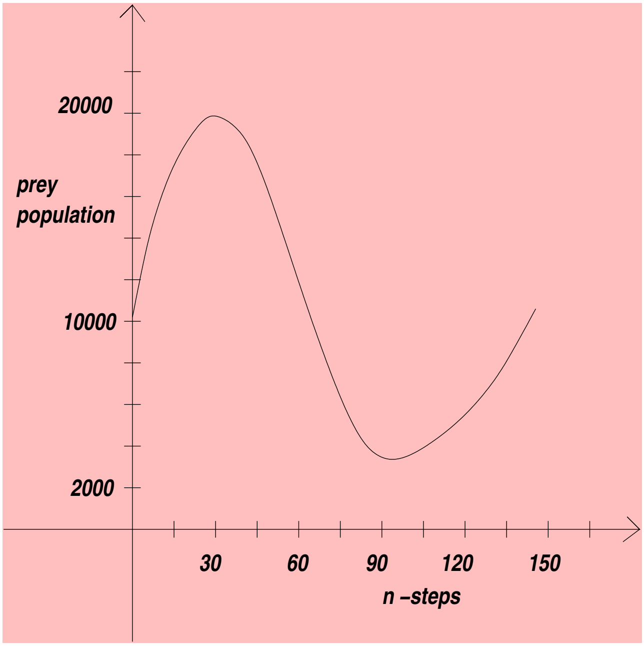


Figure 2.2 *The predator-prey solution for* $x_1 = 10000$ *and* $y_1 = 50$.

Recall the Fibonacci sequence given by the relation

$$p_n = p_{n-1} + p_{n-2}, \quad n \geq 3, \quad p_1 = 1, \quad p_2 = 1.$$

The Fibonacci sequence can also be derived from the consideration of the ancestry of a given male bee. In our discussion it is assumed that a common ancestor obtained along different paths in the ancestor tree are counted as separate ancestors. In addition, we make the following assumptions:

1. If a bee egg is laid by an unfertilized female, then it will hatch as a male.
2. If a bee egg was fertilized by a male bee, then it will hatch as a female.

In these circumstances, a male bee will always have exactly one genetic female parent. A female bee will always have exactly two genetic parents.

If one traces the ancestry of the male bee the number of ancestors at each level will give a Fibonacci sequence.

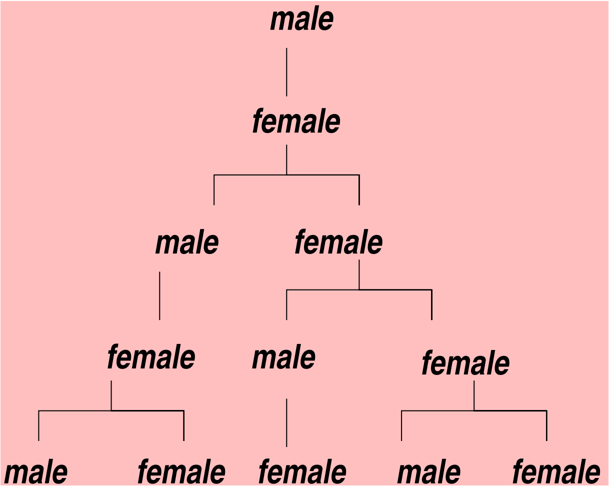


Figure 2.3 *The ancestral tree of a male bee.*

One of the interesting consequences of viewing a recursion realtion such as the Fibonacci sequence in this way is to observe that it is a result of inheritance within a simple set of rules. Many natural phenomena are also a result of some sort of inheritance over a simple set of rules. As an example, To study this sequence in more detail, we first note that

$$1 < \frac{p_n}{p_{n+1}} < 2.$$

Let $u_n = p_{n+1}/p_n$. Since the ratio is bounded and increasing as n increases (we will assume this without a specific proof) we know that it must have a limit $\lim_{n\to\infty} u_n = L$. The Fibonacci relation can be written in the form

$$1 = \frac{p_{n+1}}{p_n} - \frac{p_{n-1}}{p_n}, \quad \text{or} \quad 1 = u_n - \frac{1}{u_{n-1}}.$$

Now since $\lim_{n\to\infty} u_n = \lim_{n\to\infty} u_{n-1} = L$, we must have that L satisfies

$$1 = L - \frac{1}{L}, \quad \text{or,} \quad L^2 - L - 1 = 0.$$

The quadratic equation is easily solved for the positive root

$$L = \frac{1+\sqrt{5}}{2} \approx 1.618033989.$$

This value is known as the golden ratio.

Recursive sequences such as the Fibonacci sequence are related to L-systems which can be used by computers to generate life-like pictures of plants and organisms. These systems involve simple rules but like the Fibonacci sequence they give rise to complex structures.

Example 1.8 Suppose that a recursion relation is given by

$$q_{n+1} = \frac{1}{1+q_n}, \quad q_1 = 1/2.$$

Find the sequence and determine the limit if it exists.

Solution 1.8 The first few terms of the sequence are given by

$$\{1/2, 2/3, 3/5, 5/8, 8/13, \ldots\}.$$

It appears that the sequence is made from the ratios of consecutive elements of the Fibonacci sequence. We might guess that the limit is just the reciprocal of the golden ratio. To see this assume that $\{p_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, \ldots\}$ denotes the Fibonacci sequence. Now observe that

$$q_2 = \frac{1}{1 + 1/2} = \frac{2}{2+1} = \frac{p_3}{p_3 + p_2} = \frac{p_3}{p_4},$$

and

$$q_3 = \frac{1}{1 + q_2} = \frac{1}{1 + p_3/p_4} = \frac{p_4}{p_5}.$$

We apply induction and assume that $q_n = p_{n+1}/p_{n+2}$, then

$$q_{n+1} = \frac{1}{1 + p_{n+1}/p_{n+2}} = \frac{p_{n+2}}{p_{n+2} + p_{n+1}} = \frac{p_{n+2}}{p_{n+3}}.$$

The limit follows by applying the quotient rule for limits in Remark 1.5.5.

Example 1.9 Suppose that a continued fraction is given by the implied recursion formula

$$x = \frac{1}{2 + \frac{1}{2 + \ldots}}.$$

Find the limit.

Solution 1.9 Given the implied form of the recursion relation we write

$$L = \frac{1}{2 + L}, \quad \text{or,} \quad L^2 + 2L - 1 = 0.$$

Since we are looking for a positive root, we obtain

$$L = \frac{-2 + \sqrt{4+4}}{2} = \sqrt{2} - 1.$$

If we treat this as a recursion relation we have

$$x_{n+1} = \frac{1}{2 + x_n}.$$

The recursion relation can be solved for the limit by replacing x_{n+1} and x_n by L as we did for the continued fraction above. On the other hand, we see that the limit must be less that $1/2$ if the limit exists. We may take $x_1 = 1/2 > L$

for example. The recursion relation can be solved for the limit by replacing x_{n+1} and x_n by L as we did for the continued fraction. Then we have

$$\{\frac{1}{2}, \frac{2}{4+1}, \frac{5}{10+2}, \frac{12}{24+5}, \frac{29}{58+12}, \ldots\}.$$

Observe that $29/70 = 0.41428 \approx \sqrt{2} - 1$.

Of course this method will only compute a correct limit when such a limit actually exists. As x_n and x_{n+1} approach a common limit L, the points can be represented by consecutive points on the graph of a quadratic equation such that the points are approaching a root of that equation.

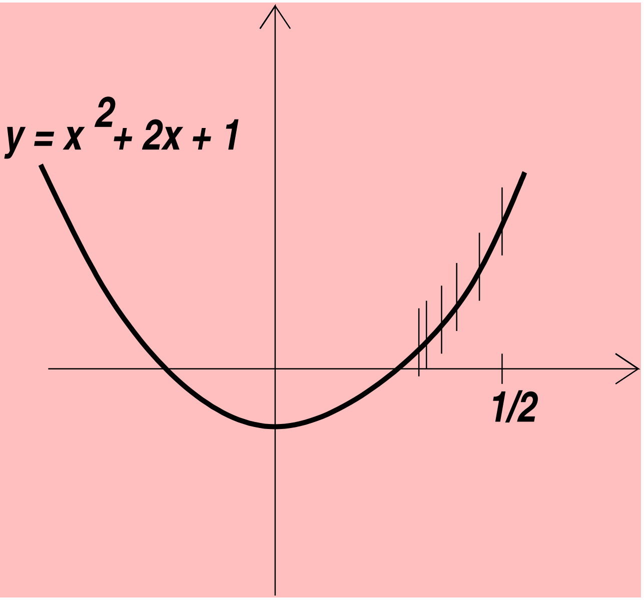


Figure 2.4 *Convergence of a continued fraction recursion along a quadratic* $x^2 + 2x - 1 = 0$.

2.2 The Idea of Infinite Series and Power Series

A *series* $\{s_n\}_{n=1}^{\infty}$ is a sequence of partial sums whose individual terms are the elements of some sequence $\{a_n\}_{n=1}^{\infty}$. We write such an element of the series in the form

$$\boxed{s_n = a_1 + a_2 + a_3 + \ldots + a_n.}$$

The series is usually represented by the expression

$$\boxed{\sum_{i=1}^{\infty} a_i, \quad \text{where} \quad s_n \; = \; \sum_{i=1}^{n} a_i.}$$

Definition.

1. We define the *harmonic series* by

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n},$$

or,

$$\sum_{i=1}^{\infty} \frac{1}{i}.$$

2. We define a *geometric series* by

$$s_n = a + ar + ar^2 + \ldots + ar^n, \quad 0 < r < 1, \; a \in \mathbf{R},$$

or,

$$\sum_{i=1}^{\infty} ar^i.$$

3. We define a *constant series* by

$$s_n = na \quad n \in \mathbf{N}, \; a \in \mathbf{R}, \quad \text{or} \quad \sum_{n=1}^{\infty} a.$$

4. We define a *p-series* by

$$s_n = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p}, \qquad 0 < p < \infty$$

or,

$$\sum_{i=1}^{\infty} \frac{1}{i^p}.$$

Idea of a Series Limit A series

$$\{\sum_{i=1}^{n} a_i\}_{n=1}^{\infty} = \{s_n\}_{n=1}^{\infty}$$

is said to converge to a limit L if we can force s_n to be as close to L as we like provided that n is chosen sufficiently large.

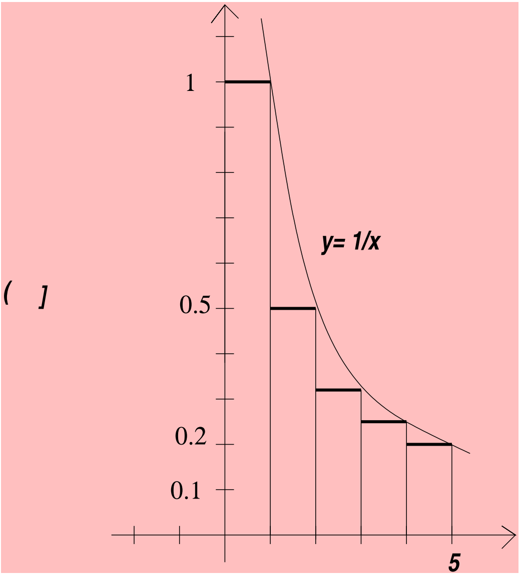


Figure 2.5 *The harmonic series can be realized as the area under a step function.*

Note that the harmonic series is a special case of the p-series.

Remark 2.2 We will use the following fact: A series $\sum a_i$ converges if and only if the sequence of partial sums $\{s_n\}_{n=1}^{\infty}$ converges under the definition for sequences.

Example 2.1 Show that the harmonic series diverges.

Solution 2.1 We observe that

$$\frac{1}{3}+\frac{1}{4}>\frac{1}{2}, \quad \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{4}{8}=\frac{1}{2}$$

and in general

$$\frac{1}{2^n+1}+\frac{1}{2^n+2}+\ldots+\frac{1}{2^n+2^n}>\frac{2^n}{2^{n+1}}=\frac{1}{2}.$$

This implies that

$$s_2 \leq \frac{3}{2}, \quad s_4 < 2, \quad s_8 < \frac{5}{2}, \ldots, s_{2^{n+1}} < \frac{3}{2}+\frac{n-1}{2}.$$

Thus the sequence of partial sums tends to infinity as n tends to infinity.

Remark 2.3 A p-series converges if and only if $p > 1$. We have shown that the p-series diverges when $p = 1$. A complete proof of this remark must wait until we have developed the theory of integration which is related to finding the areas under curves.

Definition. A power series is a sequence of polynomial functions (each polynomial represents a partial sum) defined by the formal summation

$$f(x)=\sum_{n=1}^{\infty} a_n x^n.$$

The power series function is defined wherever the summation converges for real numbers.

Definition. A Fourier series is a sequence of trigonometric functions defined by the formal summation

$$f(x)=a_o+\sum_{n=1}^{\infty} a_n \cos(\frac{\pi x}{L})+b_n \sin(\frac{\pi x}{L}).$$

A Fourier series function is defined on the domain such that the summation converges for real numbers in that domain.

Example 2.2 Show that every polynomial is a power series.

Solution 2.2 Let

$$f(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_o$$

represent some polynomial. Then there exists a power series

$$\sum_{n=1}^{\infty} b_n x^n,$$

such that

$$a_o = b_o, \quad a_1 = b_1, \quad ... \quad a_k = b_k, \quad b_n = 0, \quad n > k.$$

This power series coincides with the original polynomial.

2.1 The Definition of Limit

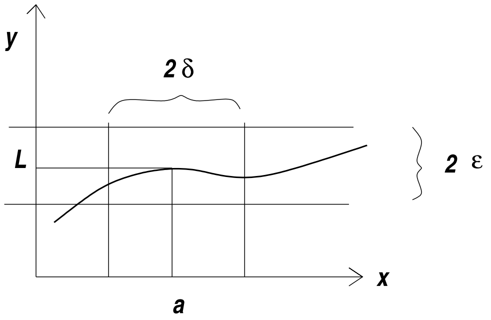


Figure 2.6 *A function with limit L at a*

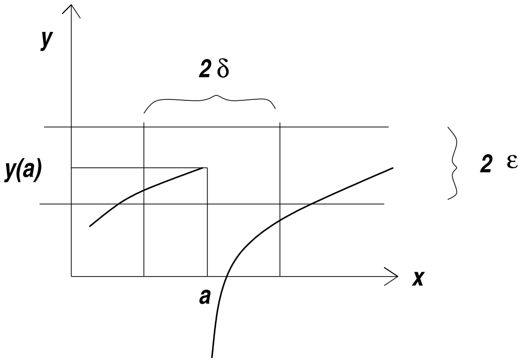


Figure 2.7 *The limit of the function does not exist at a*

Definition 2.1.1 *A function $f(x)$ defined on some open interval (α, β) is said to have a limit L at a point $a \in (\alpha, \beta)$ if for each $\epsilon > 0$ there exists some $\delta > 0$ such that*

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

The Idea of Limits *A function $f(x)$ has a limit L at a if we can force the value of the function to be very close to L by carefully choosing the interval of allowable points near a.*

2.2 Rules for Applying Limits

2.3 The Definition of Continuity

2.4 Properties of Continuous Functions

2.5 The Intermediate Value Theorem

CHAPTER 3. THE THEORY OF DIFFERENTIATION

3.1 The Definition of Derivative

3.2 The Power Rule for Differentiation

3.3 The Product and Quotient Rules

3.4 The Chain Rule for Differentiation

3.5 Differentiation of Trigonometric Functions

3.6 The Mean Value Theorem

3.7 Maximum and Minimum Values for Functions

Chapter 4. Applications

4.1 Anti-derivative

4.2 Logarithmic Functions

4.3 Exponential Functions

4.4 Population Dynamics

4.5 Radiative Decay and Pollution

CHAPTER 5. INTEGRATION

5.1 Riemann Sums

The technique for replicating the sound waves is as follows

1. A specific length of time is fixed to sample the sound wave, or equivalently the number of samples per second is set.
2. Each sample is described on a coordinate axis using time and amplitude where the amplitude at each point is given by a pressure-density reading.
3. Each sample is treated as a continuous function on a closed interval with independent variable t.
4. A finite representation of the Fourier series is used to approximate the function on that interval.

5.2 Indefinite Integrals

5.3 Area as a Limit of Sums

5.4 Signed Areas

5.5 The Definite Integral

5.6 The Fundamental Theorem of Calculus

CHAPTER 6.
APPLICATIONS

7.1 Difference Equations

7.2 Epidemic Growth

7.3 The Dooms Day Problem

7.4 The Predator-Prey Problem

7.5 Volumes of Solids

7.6 Integral Geometric Formulas

CHAPTER 7.
POWER SERIES

6.1 The Definition of a Power Series

6.2 Basic Results for Power Series

6.3 The Maclaurin Series

6.4 The Taylor Series