Mathematics 236

The Big-M Method:

What if you cannot find an initial Basic Feasible Solution?

(Reference: Winston, fourth ed., Section 4.12)

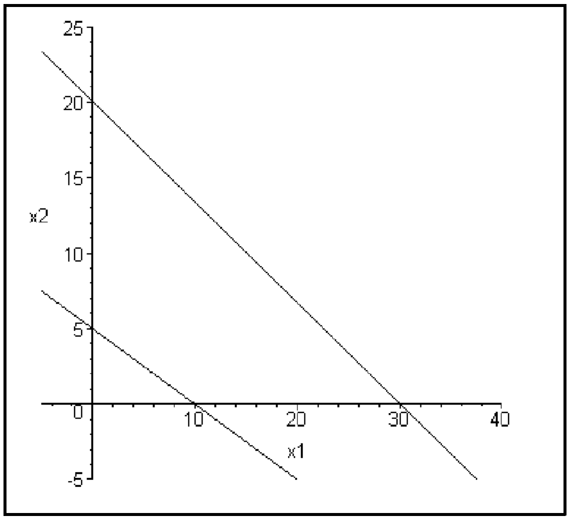

Figure 1

An example to illustrate the difficulty:

Suppose we had the linear programming problem of minimizing z = $4x_1 + 5x_2$ subject to the constraints (1) $x_1 + 2x_2 \geq 10$; (2) $2x_1 + 3x_2 \leq 60$; and x_1, x_2 both nonnegative. The feasible region is shown in Figure 1. Note that the origin is *not* in the feasible region.

Suppose we start to solve this using the Simplex Algorithm: we would convert the problem to one in standard form, rewriting the constraints as

(1) $x_1 + 2x_2 - e_1 = 10$, and

(2) $2x_1 + 3x_2 + s_2 = 60$,

with all of x_1, x_2, e_1, and s_2 nonnegative.

Next, we would set up the data table:.

z	x_1	x_2	e_1	s_2	RHS
1	-4	-5	0	0	0
0	1	2	-1	0	10
0	2	3	0	1	60

Note that we cannot get an initial basic feasible solution (because the coefficient of e_1 is -1, not +1). If we made it a +1 be multiplying Row 1 (or constraint 1) by -1, the RHS would become -10, and this is no good. (ALL RHS's must be ≥ 0 for the Simplex Algorithm to work.)

We're stuck! Remember the very first step in the Simplex Procedure was "Get an initial basic feasible solution."

The way we get around this problem is to "expand the feasible region" by embedding it into a higher dimensional space. This is done by introducing one or more **Artificial Variables** – these come from new, additional dimensions that are purely figments of our imagination. We introduce one such variable for every constraint that causes us difficulty.

In the present case, the "problem constraint" is the first one, $x_1 + 2x_2 \geq 10$; so we introduce the artificial variable A_1, changing the constraint to

(1 revised) $x_1 + 2x_2 + A_1 \geq 10$, with $A_1 \geq 0$ also.

After having done this, our (revised) problem is:

Mimimize $z = 4x_1 + 5x_2$, subject to the constraints

(1) $x_1 + 2x_2 + A_1 \geq 10$; (2) $2x_1 + 3x_2 \leq 60$; with all of x_1, x_2 , A_1 nonnegative.

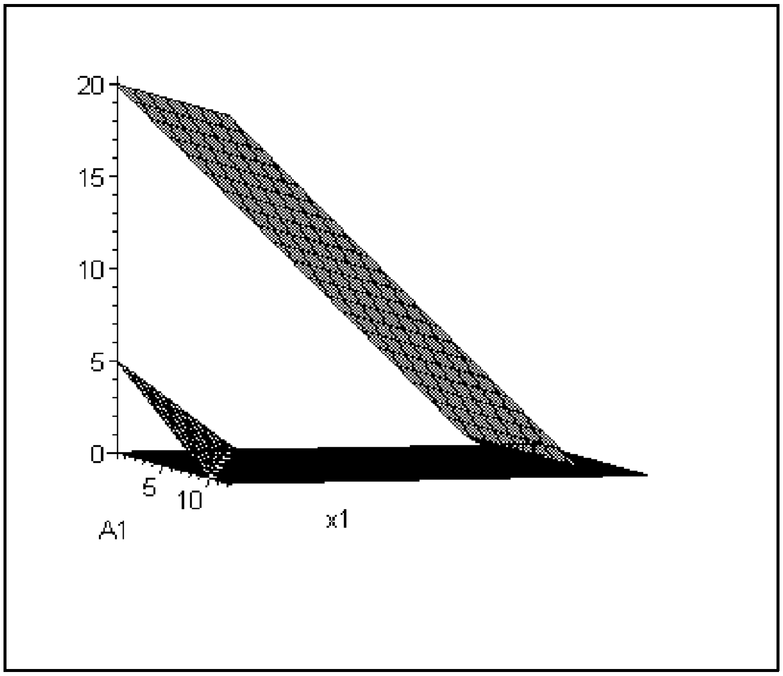

Figure 2: Expanded Feasible Region

You can think of our **"Expanded Feasible Region"** as being in 3-d space. (Figure 2) It is the 3-dimensional region bounded by the two planes coming towards us; the "back wall (where $A_1 = 0$) and the floor (where $x_2 = 0$) .

The "Real Feasible Region" is the part of the expanded region that is ON the back wall.

Let's try to solve this using the Simplex Method. Convert to standard form (introduce e_1 and s_2); and then set up our data table. We get

z	x_1	x_2	A_1	e_1	s_2	RHS
1	-4	-5	0	0	0	0
0	1	2	1	-1	0	10
0	2	3	0	0	1	60

This *is* a "proper Simplex Tableau": the two basic variables are A_1 and s_2. Moreover, since the goal is to MINIMIZE z, this appears to be an optimal solution. The solution is $x_1 = x_2 = 0$; $A_1 = 10$, $e_1 = 0$ (it's non-basic) and $s_2 = 60$. Here, $z = 0$.

The only problem with this is that ***it is not TRULY a feasible solution.*** The point where $x_1 = x_2 = 0$ is not part of the feasible region for our "real problem".

What's gone wrong? It is this: we have introduced this artificial dimension, and we know that when $A_1 = 0$, we are in the "real feasible region". However, our stated goal of minimizing $4x_1 + 5x_2$ includes NO INCENTIVE for us to make A_1 equal to zero. There is no reason, in this "expanded problem", for us to try to get to the back wall (the real feasible region).

The way we will deal with this is ***to provide an incentive*** for A_1 to be zero. Or, to be more precise, we'll levy a tax, a PENALTY, if we are at a place where A_1 is non-negative. ***The more non- negativc A_1 is, the bigger the penalty will be.***

We revise the objective function, so our goal becomes

$$\text{Minimize } z = 4x_1 + 5x_2 + \text{(penalty term)} = 4x_1 + 5x_2 + MA_1$$

where M represents a “Monstrously Large Positive Number”. How big will M be? Think of how many hamburgers MacDonald’s has made. Think of the deficit of the USA. Multiply the two – if we took M to be that value, it should be big enough to make us want A_1 equal to zero. If not, we’ll take M to be even bigger.

We don’t need to specify in advance what the value of M is: just think of it as a “Monstrously Large Positive Number”.

Note: we want to make sure the penalty for having A_1 not zero is truly a penalty and not a reward. Therefore, we must MAKE SURE THE PENALTY TERM works *opposite* to the goal we seek.

For example, if the problem had been to MAXIMIZE $z = 4x_1 + 5x_2$, we don’t want to *add* MA_1, we’d want to *subtract it* making our revised goal

$$\text{Maximize } z = 4x_1 + 5x_2 - MA_1$$

Returning to the problem at hand, it is now
Minimize $z = 4x_1 + 5x_2 + MA_1$
subject to the constraints
(1) $x_1 + 2x_2 + A_1 \geq 10$; and (2) $2x_1 + 3x_2 \leq 60$, with all of x_1, x_2, A_1 nonnegative.

Let’s try to solve this problem using the Simplex Algorithm.

As usual, we would put it in standard form (introduce excess, slack variables as needed); then set up the initial data table. This is what we get ....

z	x_1	x_2	A_1	e_1	s_2	RHS	Note that this is not a “Proper Simplex tableau”. We want A_1 and s_2 to be basic variables, so we need a zero in the z-row, in each of these columns. To get this, we ...
1	-4	-5	-M	0	0	0	
0	1	2	1	-1	0	10	
0	2	3	0	0	1	60	

adjust the z row: Add multiples of the CONSTRAINT rows to the z-row to get 0 in the A_1 column.

Here, the operation we need to do is Row 0 <------- Row 0 + M(Row 1)

1	-4+M	-5+2M	0	-M	0	10 M	Adjusted row
0	1	$\boxed{2}$	1	-1	0	10	same
0	2	3	0	0	1	60	same

This is not optimal (we are minimizing), so we pivot. The pivot column is the x_2 column; (remember that M = BIG POSITIVE number). The row ratios are 10/2 and 60/3; the pivot row is Row 1.

The boxed element marked is the pivot element. Now we pivot.

The operations we have to do are
Row 0 <— (Row 0) + (5-2M)/2 (Row 1) (ugghhh !! - but do not panic!)
Row 1 <— (½)(Row 1)
Row 2<— (Row 2) - (3/2)(Row 1)

z	x_1	x_2	A_1	e_1	s_2	RHS
1	- 3/2*	0	5/2-M	- 5/2	0	25*
0	1/2	1	1/2	-1/2	0	5
0	1/2	0	-3/2	3/2	1	45

* details:
In the x_1 column:
-4+M +(5-2M)/2 *(1)
= -4+M +5/2 - M = (-8+5)/2 = - 3/2
and in the RHS column:
10 M +(5-2M)/2*(10) =
10M +25 - 10M = 25

This is optimal. The basic variables here are x_2 and s_2 . The optimal solution to our problem is $x_1 = 0$ (non basic); $x_2 = 5$; $A_1 = e_1 = 0$ (non-basic); and $s_2 = 45$. The minimum attainable z-value is z= 25.

Check your work!! Are these numbers consistent with the original equations?
(1) Is $x_1 + 2x_2 + A_1 - e_1 = 10$? LHS = 0+ 2(5) + 0 - 0 = 10 = RHS. Good!
(2) $2x_1 + 3x_2 + s_2 = 60$? LHS = 2(0) + 3(5) + 45 = 60 = RHS. Good again!

Finally, does the z-value work out? At this point, $z = 4x_1 + 5x_2 - MA_1 = 4(0) + 5(5)-M(0) = 25$.

If these had NOT worked out, we'd know we'd made an error somewhere.

When do we NEED to introduce artificial variables?

We need to introduce a new artificial variable or variables in two cases:

- whenever we have a $\geq$ constraint (with nonnegative RHS). As in the example above, we needed to replace $x_1 + 2x_2 \geq 10$ by $x_1 + 2x_2 + A_1 \geq 10$.
- whenever we have an **equality** constraint. For example, if we had the constraint $7x_1 + 8x_2 - 6x_3 = 16$; we'd to replace it by $7x_1 + 8x_2 - 6x_3 + A' = 16$, where A' is a new artificial variable.

Of course, we'd need to add a penalty term to z, the objective function, for each artificial variable. For example:

- Maximizing: new z = (old z) minus (penalty term) = (old z) - $MA_1 - MA_2 - \ldots$.
- Minimizing: new z = (old z) plus (penalty term) = (old z) + $MA_1 + MA_2 + \ldots$.

Summary of the Big M Method (compare with page 174, Winston)

- introduce a new artificial variable for each $\geq$ constraint and each equality constraint. Make sure ALL these new variables are nonnegative, too;
- modify the objective function to include the *right kind* of penalty term for each artificial variable introduced;
- convert the (revised) problem to standard form; then set up the initial data table;
- in your initial simplex tableau, EACH artificial variable will be basic. (Some others may be basic, too.)
- DON'T FORGET to adjust the z-row to get 0 in the z-row in each column coming from an artificial variable.

Once all these preparations have been done, proceed as normal.

NOTE - initially all the artificial variables will be basic. Most of the time, the initial pivots will (one by one) choose an artificial variable as the pivot column. After pivoting, this artificial variable will become non-basic (and so have the value = zero at the current BF solution).

Once an artificial variable becomes non-basic, there should NEVER be a reason to bring it back into the basis. THEREFORE, it will never become non-zero again. We can forget about it. ONCE AN ARTIFICIAL VARIABLE LEAVES THE BASIS IT IS SAFE TO DELETE THAT VARIABLE FROM THE PROBLEM. Just cross out the column for that variable.

Computational Note for Hand Calculations

As we saw in the example above, hand calculations get messy when the objective row has “M-terms”. A way to make the work less cumbersome is to ***split the z-row*** into two parts: a non-M part and an M-part, and work with each part separately.

Here’s how to re-do the calculations this way. Our first data table is

z	x_1	x_2	A_1	e_1	s_2	RHS	We could just leave the 0M terms blank
1	-4	-5	0	0	0	0	
+0M	+0M	+0M	-M	+0M	+0M	+0M	Again, this is not a “Proper Simplex tableau”. We want A_1 and s_2 to be basic variables, so we need a zero in the z-row, in each of these columns. To get this, we ...
0	1	2	1	-1	0	10	
0	2	3	0	0	1	60	

Adjust the z row: Add multiples of the CONSTRAINT rows to the z-row to get 0 in the A_1 column.

Here, the operation we need to do is Row 0 <------- Row 0 + M(Row 1)

1	-4	-5	0		0		Row 0
	+M	+2M		-M		10 M	+ M(Row 1)
0	1	$\boxed{2}$	1	-1	0	10	same
0	2	3	0	0	1	60	same

This is not optimal (we are minimizing), so we pivot. The pivot column is the x_2 column; (remember that M = BIG POSITIVE number); The row ratios are 10/2 and 60/3; the pivot row is Row 1.

The boxed element the pivot element. Now we pivot. The operations we perform are:

Row 0 <— (Row 0) + (5/2)(Row 1) -(M)(Row 1)
Row 1 <— (½)Row 1
Row 2<— (Row 2) - (3/2)(Row 1)

z	x_1	x_2	A_1	e_1	s_2	RHS	Work on the separate parts of Row 0:
1	- 3/2	0	5/2	- 5/2	0	25	Row 0 +(5/2) (Row 1) ...
			-M				... -M(Row 1)
0	1/2	1	1/2	-1/2	0	5	
0	3/2	0	-3/2	3/2	1	45	

Again, this is optimal (since M>>0). The solution is $x_1 = 0$ (non basic); $x_2 = 5$, and the optmal z-value is z= 25. The other variables have values $e_1 = 0$ (non-basic); $s_2 = 45$ and $A_1 = 0$. Because A_1 is zero, this solution is “truly feasible”.

Worked Example: The Bevco Problem (Pages 172-177, Winston text.)

Your job is to fill in the blank spaces in the tableaux below.

Original Problem:	Revised Problem in Standard Form:
Minimize $z = 2x_1 + 3x_2$ subject to $(1/2)\, x_1 + (1/4)\, x_2 \leq 4$ $x_1 + 3x_2 \geq 20$ $x_1 + x_2 = 10$ and x_1, x_2 nonnegative	Minimize $z = 2x_1 + 3x_2 + MA_2 + MA_3$ subject to $(1/2)\, x_1 + (1/4)\, x_2 + s_1 = 4$ $x_1 + 3x_2 - e_2 + A_2 = 20$ $x_1 + x_2 + A_3 = 10$ and x_1, x_2, e_2 , A_2 , A_3 nonnegative

Our first data table. We can avoid having to recopy the constraint rows.

z	x_1	x_2	s_1	e_2	A_2	A_3	RHS	
1	-2	-3	0	0	0 -M	0 -M	0	Original Row 0
1			0		0	0		Revised Row 0 = Row 0 ... ... +M(Row 2) + M(Row 3)
0	1/2	1/4	1	0	0	0	4	
0	1	3**	0	-1	1	0	20	
0	1	1	0	0	0	1	10	

This isn't optimal. (Why not?)
The pivot column is the ___________________ Column.

And the row ratios are ______, ________, and ______.

The pivot row is Row _________________ .

After our first pivot, the tableau is ...

			Row 0 + Row 2 - (4/3)M Row 2
			Row 1 -(1/4)(1/3) Row 2
			(1/3) Row 2
			Row 3 - (1/3) Row 2

continued ....

Your answer should be this:

z	x_1	x_2	s_1	e_2	A_2	A_3	RHS	
1	-1 + 2/3 M	0	0	-1 1/3 M	1 -4/3 M	0	20 +10/3 M	Row Ratios
0	5/12	0	1	1/12	-1/12	0	7/3	
0	1/3	1	0	-1/3	1/3	0	20/3	
0	2/3	0	0	1/3	-1/3	1	10/3	

Note the following.

- z has improved (decreased) from 30M to 20 + 10/3 M
- A_2 is now a non-basic variable, so we could delete the A_2 column. That's why it's shaded.

This isn't optimal. Do you see why?

The pivot col is the ______________ column. (We are minimizing so we want the most POSITIVE / NEGATIVE z-row coefficient.)

The pivot row is ________________________

Our next tableau is ...

1		0	0	-1/2	1/2 - M			Row 0 + 3/2(Row 3) -M (Row 3)
0	0	0	1		1/8			Row 1 -(5/12)(3/2)Row 3 = Row 1 -5/8 Row 3
0	0	1	0	-1/2	1/2	-1/2	15/3	Row 2 - (½) Row 3
0	1	0	0	1/2	-1/2	3/2	5	(3/2) Row 3

The top row would be read as

z	x_1	x_2	s_1	e_2	A_2	A_3	RHS
1	0	0	0	-1/2	½ -M	3/2 -M	25

All the z-row coefficients (except z's) are negative or zero, so we are at an optimal solution.
The optimal z-value is 25, and it is attained at the place where $x_1 = 5$ and $x_2 = 5$.