Student Responsibilities — Week 4

- **Reading**: Textbook, Section 2.1–2.3

- **Assignments**:
  - Sec 2.1 Due **Wed**: 2ab, 4, 10, 12, 16, 18, 19c, 22, 24
  - Sec 2.2 Due **Thur**: 2, 4, 6, 8, 16abc, 26ab, 32, 48a, 53
  - Sec 2.3 Due **Mon**: 2, 8, 10, 12, 16, 28, 34, 48

- **Attendance**: Strongly Encouraged

**Week 4 Overview**

- 2.1 Sets
- 2.2 Set Operations
- 2.3 Functions
2.1 Sets

- **Set**: an *unordered* collection or group of objects, which are said to be *elements*, or *members* of the set

- A set is said to *contain* its elements

- There must be an underlying *Universal Set, \( U \)*, either specifically stated or understood
Notation used to specify sets

- **list the elements between braces**; listing an object more than once does *not* change the set—ordering means nothing.

  \[
  S = \{a, b, c, d\} = \{b, c, a, d, d\}
  \]

- **Set builder notation** – specify by predicate; here, \(S\) contains all elements from \(U\) which make the predicate \(P\) true

  \[
  S = \{ x \mid P(x) \}
  \]

- **brace notation with ellipses**; here, the negative integers:

  \[
  S = \{\ldots, -3, -2, -1\}
  \]
Common Universal Sets

- $\mathbb{R}$ — Real Numbers

- $\mathbb{N}$ — Natural Numbers: \{0, 1, 2, 3, \ldots \}

- $\mathbb{Z}$ — Integers: \{\ldots, −3, −2, −1, 0, 1, 2, 3, \ldots \}

- $\mathbb{Z}^+$ — Positive Integers

- $\mathbb{Q}$ — Rational Numbers: \{ $\frac{p}{q}$ | $p, q \in \mathbb{Z} \land q \neq 0$ \}
Notation

- \( x \in S \) — \( x \) is a member of \( S \), or \( x \) is an element of \( S \)

- \( x \not\in S \) — \( x \) is not an element of \( S \)

Set Equality — Definition #1

- Two sets are **equal** if and only if they have the same elements.

- That is, if \( A \) and \( B \) are sets, then \( A \) and \( B \) are equal if and only if \( \forall x [x \in A \leftrightarrow x \in B] \).

- We write \( A = B \) if \( A \) and \( B \) are equal sets.
Subsets

- **Subset**: Let $A$ and $B$ be sets. Then
  \[ A \subseteq B \iff \forall x \ [ x \in A \rightarrow x \in B ] \]

- **Empty, void, or Null Set**: $\emptyset$ is the set with no members
  
  - the assertion $x \in \emptyset$ is always false, thus
  \[ \forall x \ [ x \in \emptyset \rightarrow x \in B ] \] is always (vacuously) true, therefore
  $\emptyset$ is a subset of every set
  
  - **Note**: a set $B$ is always a subset of itself

- **Proper subset**: $A \subset B$ if $A \subseteq B$, but $A \neq B$
Cardinality

- **Cardinality** of a set: the number of distinct elements in the set.

- If there are exactly $n$ elements in a set $S$, where $n$ is a nonnegative integer, we say $S$ is a **finite** set and that $n$ is the cardinality of $S$.

- The cardinality of $S$ is denoted $|S|$.

- A set is said to be **infinite** if it is not finite.
Power Set

- **Power Set:** \( \mathcal{P}(A) \) is the set of *all* possible subsets of the set \( A \)

- If \( A = \{a, b\} \), then
  \[ \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \]

- What is the power set of the set \( B = \{0, 1, 2\} \)?

- How many elements would \( \mathcal{P}(\{a, b, c, d\}) \) have?
Cardinality

- **Cardinality**: $|A|$ is the number of distinct elements in $A$

- If the cardinality is a natural number (in $\mathbb{N}$), then the set is called **finite**; otherwise, it’s called **infinite**

- **Example**: Let $A = \{a, b\}$
  - $|\{a, b\}| = 2$
  - $|\mathcal{P}(A)| = |\mathcal{P}(\{a, b\})| = 4$
  - $A$ is finite, and so is $\mathcal{P}(A)$

- **Note**: $|A| = n \rightarrow |\mathcal{P}(A)| = 2^n$

- **Note**: $\mathbb{N}$ is infinite since $|\mathbb{N}|$ is not a natural number — it is called a **transfinite cardinal number**

- **Note**: Sets can be both **members** and **subsets** of other sets
Example

Let $A = \{\emptyset, \{\emptyset\}\}$

- $A$ has two elements and hence four subsets:
  - $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}\}$
- Note that $\emptyset$ is both a member of $A$ and a subset of $A$

Russell’s Paradox: Let $S$ be the set of all sets which are not members of themselves.

Is $S$ a member of itself or not?

The Paradox of the Barber of Seville: The (male) barber of Seville shaves all and only men who do not shave themselves. Who shaves the barber of Seville?
Cartesian Product

- **Cartesian Product of A with B**: $A \times B$ is the set of ordered pairs: 
  \[
  \{ <a, b> \mid a \in A \land b \in B \}\]

- **Notation**: \[\prod_{i=1}^{n} A_i = \{ <a_1, a_2, \ldots, a_n> \mid a_i \in A_i \},\]
a n–tuple

- The Cartesian product of any set with $\emptyset$ is $\emptyset$ — why?

- **Example 1.** Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$
  \[
  A \times B = \{ <a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3> \}
  \]
  What is $B \times A$?

  If $|A| = m$ and $|B| = n$, what is $|A \times B|$?
The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$ is the set of ordered $n$–tuples $< a_1, a_2, \ldots, a_n >$, where $a_i \in A_i$ for $1 \leq i \leq n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{ < a_1, a_2, \ldots, a_n > | a_i \in A_i, i = 1, 2, \ldots, n \}$$

If $A = \{a, b\}$ and $B = \{1, 2, 3\}$, what is $A \times B \times A$?
Quantifiers

We can use quantifiers to restrict the domain

- $\forall x \in S[P(x)]$ denotes $\forall x [x \in S \rightarrow P(x)]$

  **Ex:** $\forall x \in \mathbb{R}[x^2 \geq 0]$ means:
  
  for every real number $x$, $x^2$ is non-negative

- $\exists x \in S[P(x)]$ denotes $\exists x [x \in S \land P(x)]$

  **Ex:** $\exists x \in \mathbb{Z}[x^2 = 1]$ means:
  
  there exists an integer $x$ such that $x^2 = 1$
Truth Sets

- Let $P$ be a predicate and $D$ a domain. The **Truth Set** of $P$ is the set of elements $x \in D \ni P(x)$ is true.

- The truth set of $P(x)$ is denoted: $\{x \in D | P(x)\}$

- Assume the domain is the set of integers. What are the truth sets:
  - $P = \{x \in \mathbb{Z} \mid |x| = 1\}$  
    - Truth Set:
  - $Q = \{x \in \mathbb{Z} \mid x^2 = 2\}$  
    - Truth Set:
  - $R = \{x \in \mathbb{Z} \mid |x| = x\}$  
    - Truth Set:

- **Note**: $\forall x P(x)$ is true over the domain $U$ IFF the truth set of $P$ is $U$.

- **Note**: $\exists x P(x)$ is true over the domain $U$ IFF the truth set of $P \neq \emptyset$. 
2.2 Set Operations

- **Boolean Algebra**: an algebraic system, instances of which are propositional calculus and set theory.

- The operators in set theory are defined in terms of the corresponding operator in propositional calculus.

- As before, there must be a universe, $U$, and all sets are assumed to be subsets of $U$. 

Equality of Sets

By a previous logical equivalence, we have:

\[ A = B \]

IFF

\[ \forall x [(x \in A \rightarrow x \in B) \land (x \in B \rightarrow x \in A)] \]

— or another definition —

\[ A = B \text{ IFF } A \subseteq B \text{ and } B \subseteq A \]
Set Operations

- **Union** of $A$ and $B$, denoted $A \cup B$, is the set
  \[ \{ x \mid x \in A \lor x \in B \} \]

- **Intersection** of $A$ and $B$, denoted $A \cap B$, is the set
  \[ \{ x \mid x \in A \land x \in B \} \]

  If the intersection is void, $A$ and $B$ are said to be **disjoint**

- **Complement** of $A$, denoted $\overline{A}$, is the set
  \[ \{ x \mid \neg(x \in A) \} = \{ x \mid x \notin A \} \]
More Set Operations

1. **Difference** of $A$ and $B$, or the **complement of $B$ relative to $A$**, denoted $A - B$, is the set

   \[ A \cap \overline{B} \]

   Note: The absolute complement of $A$ is $U - A$

2. **Symmetric Difference** of $A$ and $B$, denoted $A \oplus B$, is the set

   \[ (A - B) \cup (B - A) \]
Examples

\[ U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \]

\[ A = \{1, 2, 3, 4, 5\} \quad \text{and} \quad B = \{4, 5, 6, 7, 8\} \]

\[ A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\} \]

\[ A \cap B = \{4, 5\} \]

\[ \overline{A} = \{0, 6, 7, 8, 9, 10\} \]

\[ \overline{B} = \{0, 1, 2, 3, 9, 10\} \]

\[ A - B = \{1, 2, 3\} \]

\[ B - A = \{6, 7, 8\} \]

\[ A \oplus B = \{1, 2, 3, 6, 7, 8\} \]
Venn Diagrams

Venn Diagrams are a useful geometric visualization tool for 3 or fewer sets.

- The Universal set $U$ is a rectangular box
- Each set is represented by a circle and its interior
- All possible combinations of the sets must be represented
- Shade the appropriate region to represent the given set operation
For 2 sets

For 3 sets
Examples

\[ A \cap B \]

\[ A \cup (C \cap B) \]
Set Identities

Set identities correspond to the logical equivalences.

Example

The complement of the union is the intersection of the complements:

\[ \overline{A \cup B} = \overline{A} \cap \overline{B} \]

To prove this statement, we need to show:

\[ \forall x \ [x \in \overline{A \cup B} \iff x \in \overline{A} \cap \overline{B}] \]

To show two sets are equal

- we can show for all \( x \) that \( x \) is a member of one set IFF it is a member of the other, or
- show that each is a subset of the other
Universal Instantiation

We now apply an important rule of inference called

**Universal Instantiation**

In a proof, we can eliminate the universal quantifier which binds a variable if we do not assume anything about the variable other than it is an arbitrary member of the Universe. We can then treat the resulting predicate as a proposition.

We say, “Let $x$ be arbitrary.” Then we can treat the predicates as propositions.
<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in \overline{A \cup B}$</td>
<td>$x \notin (A \cup B)$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$\neg [x \in (A \cup B)]$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$\neg [(x \in A) \lor (x \in B)]$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$\neg (x \in A) \land \neg (x \in B)$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$(x \notin A) \land (x \notin B)$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$(x \in \overline{A}) \land (x \in \overline{B})$</td>
</tr>
<tr>
<td>$\iff$</td>
<td>$x \in (\overline{A} \cap \overline{B})$</td>
</tr>
</tbody>
</table>

Hence, $x \in \overline{A \cup B} \iff x \in \overline{A} \cap \overline{B}$ is a tautology since

- $x$ was arbitrary, and
- we have used only logically equivalent assertions and definitions
Universal Generalization

We can apply another rule of inference

\[ \forall x \left[ x \in \overline{A \cup B} \iff x \in \overline{A} \cap \overline{B} \right] \]

Q.E.D. — an abbreviation for the Latin phrase “Quod Erat Demonstrandum” — “which was to be demonstrated” — used to signal the end of a proof.
Alternative Identity

Note: as an alternative which might be easier in some cases, use the identity:

\[ A = B \iff [A \subseteq B \text{ and } B \subseteq A] \]

Example

Show \( A \cap (B - A) = \emptyset \)

The empty set is a subset of every set. Hence,

\[ A \cap (B - A) \supseteq \emptyset \]

Therefore, it suffices to show

\[ A \cap (B - A) \subseteq \emptyset \]

or

\[ \forall x [x \in A \cap (B - A) \rightarrow x \in \emptyset] \]

So, as before, we say “let \( x \) be arbitrary”
Now we need to show
\[ x \in A \cap (B - A) \rightarrow x \in \emptyset \]
is a **tautology**.

But the **consequent** is always **false**.

Therefore, the **antecedent** (or premise) must also be **false**.
We proceed by applying the definitions:

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \in A \cap (B - A)$</td>
<td>Defn of intersection</td>
</tr>
<tr>
<td>$\iff (x \in A) \land [x \in (B - A)]$</td>
<td></td>
</tr>
<tr>
<td>$\iff (x \in A) \land [(x \in B) \land (x \notin A)]$</td>
<td>Defn of difference</td>
</tr>
<tr>
<td>$\iff [(x \in A) \land (x \notin A)] \land (x \in B)$</td>
<td>Comm Prop of AND</td>
</tr>
<tr>
<td>$\iff F \land (x \in B)$</td>
<td>Table 6 in textbook</td>
</tr>
<tr>
<td>$\iff F$</td>
<td>Domination</td>
</tr>
</tbody>
</table>

Hence, because $P \land \neg P$ is always false, the implication is a tautology. The result follows by Universal Generalization. Q.E.D.
Indexed Collections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets. Union and intersection are associative (because \texttt{AND} and \texttt{OR} are), we have:

\[
\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n
\]

and

\[
\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n
\]

Examples

Let $A_i = [i, \infty)$, $1 \leq i < \infty$

\[
\bigcup_{i=1}^{n} A_i = [1, \infty)
\]

\[
\bigcap_{i=1}^{n} A_i = [n, \infty)
\]
2.3 Functions

- **Function**: Let $A$ and $B$ be sets. Then a function (mapping, map) $f$ from $A$ to $B$, denoted $f : A \rightarrow B$, is a subset of $A \times B$ such that
  \[
  \forall x \ [x \in A \rightarrow \exists y \ [y \in B \land \langle x, y \rangle \in f]]
  \]
  and
  \[
  [\langle x, y_1 \rangle \in f \land \langle x, y_2 \rangle \in f] \rightarrow y_1 = y_2
  \]

- **Note**: $f$ associates with each $x$ in $A$ **one and only one** $y$ in $B$.

- $A$ is called the **domain** of $f$

- $B$ is called the **codomain** of $f$
If $F(x) = y$:

- $y$ is called the **image** of $x$ under $f$
- $x$ is called a **preimage** of $y$

Note: there may be more than one preimage of $y$, but there is only **one** image of $x$.

The **range** of $f$ is the set of all images of points in $A$ under $f$; it is denoted by $f(A)$.
Injections, Surjections, and Bijections

Let $f$ be a function from $A$ to $B$

- **Injection**: $f$ is **one–to–one** (denoted 1–1) or **Injective** if preimages are unique
  
  Note: this means that if $a \neq b$ then $f(a) \neq f(b)$

- **Surjection**: $f$ is **onto** or **surjective** if every $y$ in $B$ has a preimage
  
  Note: this means that for every $y$ in $B$ there must be an $x$ in $A$ such that $f(x) = y$

- **Bijection**: $f$ is **bijective** if it is surjective and injective, in other words, **1–1 and onto**.
Example 1

1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?
4. How do we determine the answers?
Example II

If $S$ is a subset of $A$, then $f(S) = \{f(s) | s \in S\}$

- $f(a) = Z$
- The image of $d$ is $Z$
- The domain of $f$ is $A = \{a, b, c, d\}$
- the codomain is $B = \{X, Y, Z\}$
- the range, $f(A) = \{Y, Z\}$
- the preimage of $Y$ is $b$
- the preimages of $Z$ are $a$, $c$, and $d$
- $f(\{c, d\}) = \{Z\}$
Example III

1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?
1. Is this an injection?
2. Is this a surjection?
3. Is this a bijection?
Cardinality

- **Note**: whenever there is a bijection from $A$ to $B$, the two sets must have the same number of elements or the same cardinality.

- That will become our definition, especially for infinite sets.

Let $A = B = \mathbb{R}$, the reals.

Determine which are injections, surjections, bijections:

<table>
<thead>
<tr>
<th>Function</th>
<th>Injection?</th>
<th>Surjection?</th>
<th>Bijection?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x) = x$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(x) = x^3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$f(x) =</td>
<td>x</td>
<td>$</td>
<td></td>
</tr>
</tbody>
</table>
Let $E$ be the set of even nonnegative integers, \{0, 2, 4, 6, \ldots \}

Then there is a bijection $f$ from $\mathbb{N}$ to $E$, the even nonnegative integers, defined by:

$$f(x) = 2x$$

Hence, the set of even nonnegative integers has the \textit{same} cardinality as the set of natural numbers.

\textbf{OH, NOES! IT CAN’T BE… E is only half as big!!}

\textbf{(But it’s TRUE.)}
Inverse Functions

**Inverse Function:** Let $f$ be a bijection from $A$ to $B$. Then the inverse of $f$, denoted $f^{-1}$, is the function from $B$ to $A$ defined as:

$$f^{-1}(y) = x \text{ IFF } f(x) = y$$

**Note:** no inverse exists unless $f$ is a bijection.
Example

Let \( f \) be defined by the diagram –

Then \( f^{-1} \) is defined as –
Inverse Applied to a Subset

- **Inverse Function over Subsets**: Let \( S \) be a subset of \( B \). Then
  \[ f^{-1}(S) = \{ x \mid f(x) \in S \} \]

- **Example**: Let \( f \) be the following function –

\[
\begin{align*}
A & \quad \xrightarrow{f} \quad B \\
\quad \begin{array}{c}
\bullet \text{a} \\
\bullet \text{b} \\
\bullet \text{c} \\
\bullet \text{d}
\end{array} & \quad \rightarrow \quad \begin{array}{c}
\bullet \text{X} \\
\bullet \text{Y} \\
\bullet \text{Z}
\end{array}
\end{align*}
\]

\[ f^{-1}(\{X, Y\}) = \{a, b\} \]
Composition

**Composition:** Let \( f : B \to C \), \( g : A \to B \).

The **composition of \( f \) with \( g \)**, denoted \( f \circ g \), is the function from \( A \) to \( C \) defined by

\[
f \circ g(x) = f(g(x))
\]

**Example:**

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B & \xrightarrow{f} & C \\
a & \xrightarrow{{}} & V & \xrightarrow{{}} & H \\
b & \xrightarrow{{}} & W & \xrightarrow{{}} & I \\
c & \xrightarrow{{}} & X & \xrightarrow{{}} & J \\
d & \xrightarrow{{}} & Y & \xrightarrow{{}} & \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{f \circ g} & C \\
a & \xrightarrow{{}} & H \\
b & \xrightarrow{{}} & I \\
c & \xrightarrow{{}} & J \\
d & \xrightarrow{{}} & \\
\end{array}
\]
Other Examples

Let \( f(x) = x^2 \) and \( g(x) = 2x + 1 \)

\[
f \circ g(x) = f(g(x)) \\
= f(2x + 1) \\
= (2x + 1)^2 \\
= 4x^2 + 4x + 1
\]

\[
g \circ f(x) = g(f(x)) \\
= g(x^2) \\
= 2x^2 + 1
\]
Discussion

- Suppose $f : B \to C$, $g : A \to B$, and $f \circ g$ is injective.

- What can we say about $f$ and $g$?

- Using the definition of injective, we know that if $a \neq b$, then $f(g(a)) \neq f(g(b))$, since the composition is injective.

- Since $f$ is a function, it cannot be the case that $g(a) = g(b)$, since $f$ would have two different images for the same point.

- Hence, $g(a) \neq g(b)$.

- It follows that $g$ must be an injection.

- However, $f$ need not be an injection... how could you show this? (counterexample)
FLOOR and CEILING Functions

- **Floor**: The FLOOR function, denoted
  \[ f(x) = \lfloor x \rfloor \text{ or } f(x) = \text{FLOOR}(x) \]
  is the largest integer less than or equal to \( x \).

- **Ceiling**: The CEILING function, denoted
  \[ f(x) = \lceil x \rceil \text{ or } f(x) = \text{CEILING}(x) \]
  is the smallest integer greater than or equal to \( x \).

- **Examples**: \( \lfloor 3.5 \rfloor = 3 \), \( \lceil 3.5 \rceil = 4 \)

- **Note**: The FLOOR function is equivalent to **truncation** for positive numbers.