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CONVEX POLYHEDRA WITHOUT SIMPLE CLOSED GEODESICS

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It is proved that, in contrast to the smooth convex surfaces homeomorphic to the two-dimensional sphere, the most of convex polyhedra in three dimensional space are free of closed geodesics without self-intersections.

1. Introduction

In 1905, in his famous article "Sur les lignes géodésiques des surfaces convexes" [1] H. Poincaré posed a question on the existence of at least three geometrically different closed geodesics without self-intersections on any *smooth* convex two-dimensional surface (2-surface) $\mathcal{M}$ homeomorphic to the two-dimensional sphere (2-sphere) S^2. Each such geodesic splits the surface $\mathcal{M}$ into two domains homeomorphic to 2-discs, and it is naturally to call it *simple geodesic*.

In 1917, G. D. Birkhoff proved [2] the existence of *at least one* closed simple geodesic on $\mathcal{M}$ (in the late 20s he extended the result to the multidimensional case [3]). Nowadays this geodesic is called the (Birkhoff) "equator". The presence of the Birkhoff equator serves as a basis for proving the existence of infinitely many (non-simple) closed geodesics on the considered surface. This very recent result has been established by V. Bangert [4] and J. Franks [5] in 1991-1992.

In 1929 L.Luysternik and L.Shnirel'man gave a proof [6] of the existence of *three* closed simple geodesics, which led to a solution of the Poincaré problem. It follows from a result of Jacobi, who integrated geodesic flows on an ellipsoid (see [7], [8]), that on an ellipsoid whose semi-axes are distinct and close to 1 there are *only three* different closed simple geodesics. Therefore, the estimate *"three"* in the theorem of Lyusternik- Shnirel'man is sharp. The complete proof of the Lyusternic-Shnirel'man theorem has been recently obtained in [9] and in [10] (see also the survey [11] on this subject, where a sketch of the proof is given).

One can raise the Poincaré question also for convex polyhedra in $\mathbf{R}^3$ homeomorphic to the 2-sphere S^2. Geodesic flows on Euclidean polyhedra is also an old subject that goes back as far as 1906 (see [12] and [13]).

It turns out that in this case, in contrast to what might be expected from what has been said above, there are convex polyhedra completely free of simple closed geodesics on their surfaces and, in a sense, such polyhedra constitute the majority in the class of all convex polyhedra (see theorems 1, 2 below). The existence of these polyhedra is due, of course, to the fact that a geodesic cannot, by definition, pass through a vertex of a polyhedron.

The situation here is the same as that for a billiard trajectory in a polygon: when reaching the vertex, the billiard particle does not "know" which direction it should move afterwards. A similar situation holds when the particle moves freely on the polyhedron surface (i.e., along a geodesic): if the particle reaches a vertex then its path is broken off and hence the trace of the particle is not deemed a

Mathematics Subject Classification 37E99, 37E35 geodesic in that case. It should be emphasized at once that closed billiard orbits (trajectories) inside a polygon can be considered as closed geodesics on a degenerate polyhedron surface. It is enough to imagine the polygon as that polyhedron possessing two identical faces (the top and bottom) and the billiard trajectory as a geodesic on that polyhedron. Namely, the billiard particle moves along a geodesic passing at each reflection from the top to the bottom of the polyhedron and vice verca. Concerning the interconnection between polygons and polyhedra we will establish at the end of this article the important property of *stability* of closed billiard orbits in a polygon [1] This property has already been established in other (combinatoric) terms in the paper [14]. At the same time, the property will show which broken lines on a generic polyhedron surface could be (non-simple) geodesics.

Some of the results of this article (with a short sketch of proofs) are published by the author in his paper [15].

2. Description of Polyhedra Free of Simple Closed Geodesics

2.1. Definition of a geodesic on the surface of a polyhedron

Let Im be a polyhedron with V vertices, E edges, and F faces, and let $\Gamma_1, \Gamma_2, ..., \Gamma_F$ be its faces. A geodesic on the surface of Im is a curve γ of locally minimal length which *does not* pass through the vertices of the polyhedron. It means that for any two points $A, B \in \gamma$ lying close to each other (the distance $\varrho(A, B)$ in $\mathbf{R}^3$ being much smaller than the diameter of polyhedron Im), the length of the part $[AB]$ of the curve γ with the end points A and B is less than the length of any other curve σ on the polyhedron surface with the same end points A, B. It is obvious that if the points A, B belong to the face Γ then $[AB]$ is the straight-line segment AB lying completely on the same face Γ (Fig. 1). If A, B are points situated on two *adjacent* faces with a common edge l, then consider the dihedral angle formed by these faces and rotate one of them around edge l until it reaches the second face's plane. Then the part of the geodesic γ connecting A and B will be (as in the first case) the segment with the end points A, B which intersects edge l at a point M (Fig. 2). Returning the rotated face into its initial position we will obtain the required part of the geodesic γ : $[AB]$ is, in this case, a broken line AMB, where the links (segments) AM and MB form *equal* angles $\alpha = \beta$ with l (Fig. 3). These angles will naturally be called the *angle of incidence* and the *angle of reflection* by analogy with those of a billiard trajectory in a polygon.

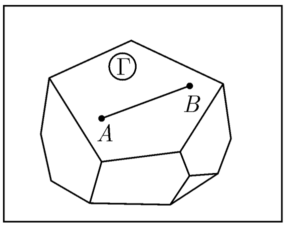

Fig. 1

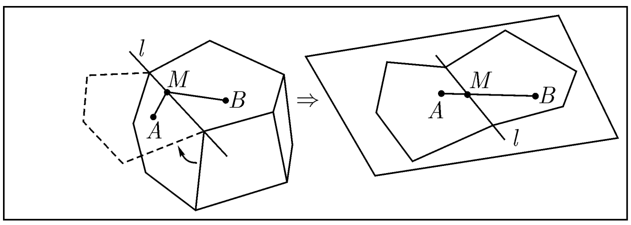

Fig. 2

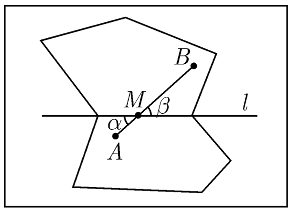

Fig. 3

This procedure of rotating the adjacent face around the common edge until it lies in the same plane as the first face if repeated infinitely many times gives a straightening of the geodesic into a line. This can be described in another way: let us fix a straight line k, dip the polyhedron into ink

[1]A closed billiard trajectory γ in a polygon is called *stable* if it is not destroyed by *any* small deformation of the polygon. In other words, by any arbitrary small change (a deformation) of the angles and sides of polygon Q to form a new polygon Q' there is a periodic tragectory $\gamma\prime$ with the same number of links as γ, and where each link is close to the correspondent link of γ.

The same definition of *stable* (with respect to any small deformation of polyhedron) closed geodesic on the polyhedron surface can be given (see part 4.1).

and roll it on the plane so that it always follows the line k. The resulting polygonal "stamps" on the plane will be the straightening of a geodesic γ on the polyhedron surface corresponding to that line k.

2.2. Problem

We shall restrict our considerations to the case of *simple* closed geodesics on a polyhedron, i.e., closed geodesics with no self-intersections. This geodesic divides the surface of the polyhedron into two parts; see Introduction. The following problem arises:

Problem: *Does there exist a simple closed geodesic on the surface of any given convex polyhedron?*

This is the problem which is solved in this paper.

2.3. The answer

Before solving the problem let us make two important remarks.

Remark 1. There is a natural topology in the set of polyhedra. Two polyhedra are close to each other if they have the same combinatoric type and their corresponding vertices as well as edges and faces are situated close to each other in space. One can calculate the number of degrees of freedom of a given polyhedron Im with V vertices, E edges, and F faces:

$$3V-6-\sum_{i=1}^{F}(k_i-3)=3V-6+3F-\sum_{i=1}^{F}k_i=(3V-6+3F)-2E=3E-2E=E\geq 6,$$

where $3V$ is the number of coordinates of $3V$ vertices; 6 is the dimension of the group of the space motions; k_i-3 parameters are needed for k_i vertices to be situated at the same plane defined by the given three of them; we register each edge twice when calculating the sum $\sum k_i$; and, finally, we take into account the Euler formula $V-E+F=2$ for a convex polyhedron.

Another way to introduce the proximity of two polyhedra Im and Im$'$ is to say that a small perturbation of Im gives Im$'$, and the points with coordinates $(\Delta_1,...,\Delta_V)$ and $(\Delta'_1,...,\Delta'_V)$ are close to each other in $\mathbf{R}^V$, where V is the number of vertices of each polyhedron and $\{\Delta_i\}$, $\{\Delta'_i\}$ are the defects of their corresponding vertices (see Definition 1 and Statement 4 below).

Remark 2. There is a natural measure in the set of all three-dimensional convex polyhedra with the same combinatorial type. Namely, every polyhedron Im $\subset\mathbf{R}^3$ whose all $S\,(=2E)$ angles $\alpha_1,...,\alpha_S$ are fixed can be represented as a point in $\mathbf{R}^S$. All such points constitute an open polyhedron $\mathcal{P}$ in $\mathbf{R}^S$, with a natural measure in $\mathbf{R}^S$. We will call the polyhedron $\mathcal{P}$ "the configuration space of all three-dimensional polyhedra of the given combinatorial type".

The (negative) answer on the posed question is given by

Theorem 1. *The class $\mathcal{P}$ of polyhedra* {Im} *homeomorphic to the sphere contains an open set of full measure of polyhedra whose surfaces are free of simple closed geodesics.*

2.4. The sharpened answer

Theorem 1 follows from Theorem 2 below which sharpens the former. Before formulating it recall the standart notion of the *defect of a vertex of a polyhedron.*

Definition 1. The defect of a vertex v *of a polyhedron* Im *is*

$$\Delta_v:=2\pi-\sum_{i\in v}\alpha_i, \tag{2.1}$$

where the $\{\alpha_i\}$ are all the plane angles of the polyhedral angle with vertex v (Fig. 4).

The sum of the vertices defects:

$$\Delta_{\mathrm{Im}} = \sum_{v\in \mathrm{Im}} \Delta_v \tag{2.2}$$

is called the defect of the polyhedron Im.

Remark. Notice that for the *convex polyhedron*, $\Delta_v > 0$ for each vertex v. One can consider the defect of a polyhedron vertex as the Gaussian curvature of a surface at a point.

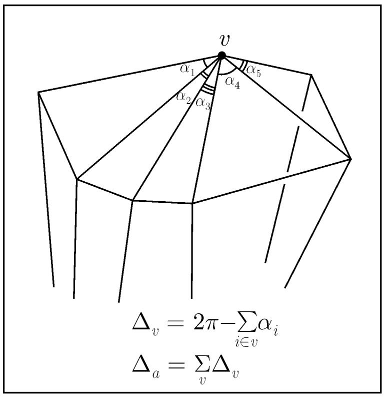

Fig. 4

Theorem 2. *Let* Im *be a polyhedron such that for any proper subset I of its vertices their defects and the number π are rationally independent:*

$$\left(\sum_{i\in I\subset\{1,2,\ldots,V\}} n_i\Delta_i + n\pi = 0;\ n_i, n \in \mathbf{Z}\right) \Rightarrow \\ \Rightarrow (n_i = n = 0,\ \forall i = 1, \ldots, V) \tag{2.3}$$

Then there are no simple closed geodesics on Im*'s surface.*

3. Proofs of the Theorems

We will give the proof of Theorem 2 in several stages.

3.1. Stage 1: defect of a polyhedron

First of all, we will need for future the following

Statement 1. (Descartes' Formula). *For any polyhedron* $\mathrm{Im} \subset \mathcal{P}$, *its defect* $\Delta_{\mathrm{Im}} = 4\pi$.

Proof. Let V, E, and F be the number of vertices, edges, and faces of the polyhedron Im, respectively. Since $\sum_{v\in\mathrm{Im}} \Delta_v = \sum_{i=1}^{V} \Delta_i$, where Δ_i is the defect of the vertex v_i, we have

$$\Delta_{\mathrm{Im}} = \sum_{i=1}^{V} \Delta_i = \sum_{i=1}^{V}(2\pi - \sum_{j\in v_i} \alpha_j^{(i)}) = 2\pi\cdot V - \sum_{i=1}^{V}\sum_{j\in v_i} \alpha_j^{(i)} = 2\pi\cdot V - \sum_{k=1}^{S} \alpha_k,$$

where S is the number of all flat angles of polyhedron Im.

Let $n_1, \ldots, n_F$ be numbers of edges of first, second, ..., F-th faces, respectively. It follows from the convexity of the polyhedron that the total number of edges of Im is half the sum of all n_i's: $\sum_{i=1}^{F} n_i = 2E$. (Notice that $\sum_{i=1}^{F} n_i = S$, and thus $S = 2E$).

Since the sum of all interior angles of the n_ith face is $\pi(n_i - 2)$, the sum of all flat polyhedron angles is equal to

$$\sum_{k=1}^{S} \alpha_k = \sum_{i=1}^{F}\sum_{\alpha_j\in n_i} \alpha_j = \sum_{i=1}^{F} \pi(n_i - 2) = \pi\cdot\sum_{i=1}^{F} n_i - 2\pi\cdot F = 2\pi(E - F).$$

Now, we will use Euler's formula $V - E + F = 2$ which holds for polyhedron homeomorphic to the sphere. We get

$$\Delta_{\mathrm{Im}} = 2\pi V - \sum_{k=1}^{S} \alpha_k = 2\pi V - 2\pi(E - F) = 2\pi(V - E + F) = 4\pi,$$

and the statement is proved. ■

Notes. **(i).** Actually, the presented proof gives a stronger result, namely: *the defect of any polyhedron, not obligatory homeomorphic to the sphere, is equal to* 2π *times its Euler characteristic* $\chi = V - E + F$.

(ii). There is another proof of the Statement 1, which had been apparently known to Descartes:

By projecting the edges of a given convex polyhedron Im from any interior point onto the unit sphere around that point, we obtain a partition of the sphere into F spherical polygons, one for each face of the polyhedron. The total angle-sum of all these polygons is clearly $2\pi V$ (i.e., 2π for each vertex); the total angle-sum of the flat faces themselves is

$$\sum_v (2\pi - \Delta_v) = 2\pi V - \Delta_{\mathrm{Im}}$$

(summed over the vertices). The difference, Δ_{Im} is the total spherical excess of the F spherical polygons, which is the total area of the spherical surface, namely 4π. ■

3.2. Stage 2: notations

Let us introduce some more notation.

Definition 2. *Let L be an arbitrary oriented closed broken line on the polyhedron* Im *such that all its break points lie on the* Im*'s edges. Rotating adjacent polyhedron faces around the common edge l_i into the same plane defines the oriented* (*with attached a sign*) *angle φ_i between two successive links of the broken line L with a common point M, as it depicted on the Fig. 5. We call this oriented angle φ_i* (*with a sign*) the defect of the break point M.

Definition 3. The defect of the closed broken line L *is the sum of all defects of break points*:

$$\delta_L := \sum_{i \in \{break\ points\ of\ L\}} \varphi_i \tag{3.1}$$

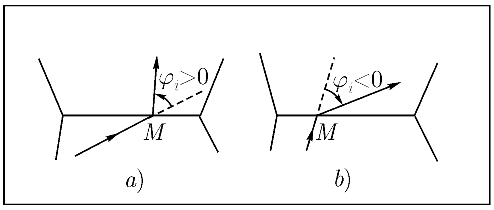

Fig. 5

It follows immediately from this definition the important but simple consequence.

Statement 2. *The defect of closed geodesic γ always equals* 0*:* $\delta_\gamma = 0$.

Proof. For this broken line γ defect of each its broken point is 0. ■

Let us define now *the index of vertex v* of the polyhedron with respect to the closed broken line L, which we denote by $ind_L v$. The introduced magnitude will be defined up to an arbitrary additional constant.

We will define at first the notion of difference of such indices, and then the index itself. To do this we connect vertices v_1, v_2 by non-self-intersecting broken line passing along the polyhedron edges. Then orient this path from the vertex v_1 to the vertex v_2, and also orient the broken line L. After that, assign "+1" in each intersection point of two curves $(v_1 - ... - v_2)$ and L, if their orientations form a positive frame (Fig. 6a), and assign "-1" if this frame is negative (Fig. 6b). Then we set, by definition,

$$\delta_L(v_1, v_2) := \sum_{k=1}^{N} (\pm 1), \tag{3.2}$$

where N is the number of intersection points of curves $(v_1 - ... - v_2)$ and L, and signs $+, -$ are set up by just shown manner. It is easy to see that this number does not depend on our choice of the path connected two given vertices.

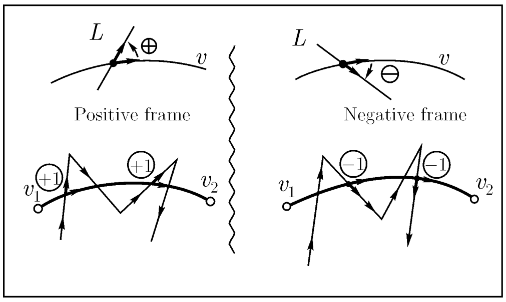

Fig. 6

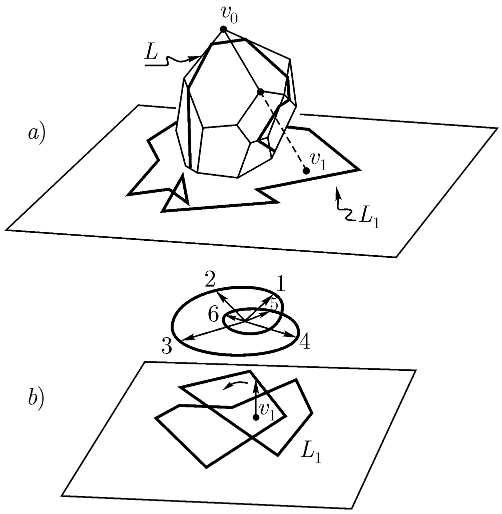

Fig. 7

Definition 4. The difference of indices *of two vertices* v_1, v_2 *with respect to the closed broken line* L*,*

$$ind_L v_1 - ind_L v_2 := \delta_L(v_1, v_2). \tag{3.3}$$

Definition 5. The index of a vertex v *with respect to the closed broken line* L *is called the sum*

$$ind_L v = ind_L v_0 + \delta_L(v_0, v), \tag{3.4}$$

where v_0 *is an arbitrary (fixed) vertex of polyhedron* Im*, and* $ind_L v_0$ *is an arbitrary number assigned to* v_0 *(usually we set* $ind_L v_0 = 0$*).*

In other words, if one projects "stereographically" from the point v_0 on any plane not parallel to any face of the polyhedron, then the broken line L will project to the broken line L_1, and the vertex v will project to the point v_1. Then $ind_L v = ind_{L_1} v_1$, where $ind_{L_1} v_1$ is the standard index of the curve L_1 in the plane of projection with respect to the point v_1 (Fig. 7a,b). Namely, it is the number of revolutions of curve L_1 around the point v_1.

The number $ind_L v$ is defined up to an (arbitrary) additive constant which is defined by the choice of the vertex v_0, or, equivalently, by the plane of "stereographic projection" (Fig. 7).

3.3. Stage 3: the Gauss-Bonnet theorem for polyhedra

Now combining all notions appearing in the definitions 1, 3, and 5; we get the analog of the Gauss-Bonnet theorem for polyhedra.

Statement 3. *For each broken line* L *on the surface of the polyhedron* Im *which does not pass through* Im*'s vertices the following holds:*

$$\delta_L = -\sum_{v_i \in \mathrm{Im}} \Delta_{v_i} \cdot ind_L v_i \pmod{2\pi}, \tag{3.5}$$

or, equivalently,

$$\delta_L + \sum_{v_i \in \mathrm{Im}} \Delta_{v_i} \cdot ind_L v_i = 2\pi k, \ k \in \mathbf{Z}. \tag{3.6}$$

Proof. For the simplicity, denote $ind_L v_i$ by $indv_i$. We will prove the relation (9). To do this, we will observe how the sum $\sum = \delta_L + \sum \Delta_{v_i} \cdot indv_i$ is changed when the curve L is retracted, or homotoped, in one of Im's faces.

When L is completely contained in a single polyhedron face, its defect δ_L is equal to $2\pi k$, and each vertex index will be 0; therefore, the sum $\delta_L + \sum \Delta_{v_i} \cdot indv_i$ is equal to $2\pi k$, and the statement follows.

The procedure of homotoping L is as follows. In the stereographic projection we see that a piece of L winds around a vertex v_i several times (see Fig. 8a). We homotop this piece of L to a curve which does not wind around the vertex v_i (see Fig. 8b), i.e., we "slide" all the loops over the vertex v_i. After this procedure the defect δ_L of the broken line decrease by the defect of the vertex Δ_{v_i} times the number of loops (Fig. 8b,c), another words, $\Delta_{v_i} \cdot indv_i$. Then we repeat again the same procedure for the other vertices.

Each time, the defect of the broken line L decreases by the defect of each vertex multiplying by its index. As a result, after all such carryings over all vertices $\{v_i\}$, δ_L will decrease by $\sum_{v_i} \Delta_{v_i} \cdot indv_i$ and be equal to $2\pi k$; therefore, the sum $\delta_L + \sum_{v_i} \Delta_{v_i} \cdot indv_i$ will not be changed and be equal to $2\pi k$. ■

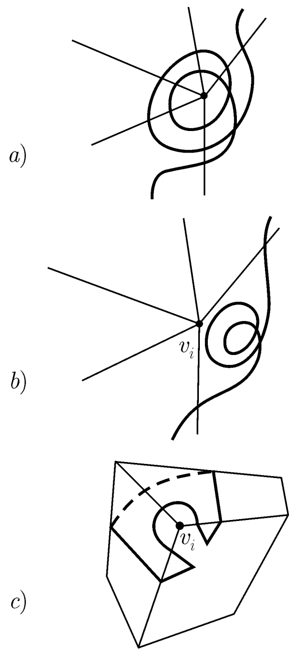

Fig. 8

3.4. Stage 4: generic tetrahedra

Here, the Theorem 2 will be proved for any generic tetrahedron.

Statement 4. (A.D. Aleksandrov, [16]). *Let $\Delta_1, ..., \Delta_n$ be arbitrary positive numbers with their sum 4π. Then there is a convex polyhedron the defect of each vertex v_i of which is egual to Δ_i, $i = 1, 2, ..., n$. In particular, if $n = 4$, then there exists a tetrahedron possessing this property.*

Statement 5. *Let T be a tetrahedron whose any proper subset of vertices' defects $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ are rationally independent with the number π. Then the tetrahedron T is free of simple closed geodesics.*

Proof. Note first of all that the equality

$$n_0\pi + n_1\Delta_1 + n_2\Delta_2 + n_3\Delta_3 + n_4\Delta_4 = 0$$

with integers $n_0, ..., n_4$ implies

$$n_1 = n_2 = n_3 = n_4, \quad n_0 = -4n_1 .$$

Indeed, since $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 4\pi$ (see Statement 1), we have

$$n_0\pi + (n_2 - n_1)\Delta_2 + (n_3 - n_1)\Delta_3 + (n_4 - n_1)\Delta_4 = 0,$$

and we get immediately from the statement hypothesis that $n_i = n_j, \forall i \neq j, 0 \leq i, j \leq 4$, and $n_0 = -4n_1$ (because of $\sum_{i=1}^{4} \Delta_i = 4\pi$).

Suppose now that there exists a simple closed geodesic γ on the T's surface. Then its defect is zero: $\delta_\gamma = 0$ (see Statement 2). It follows from the Statement 3 that

$$0 = 2\pi k - \sum_{v_i} \Delta_i \cdot ind_\gamma v_i.$$

Denoting $n_i = ind_\gamma v_i$ we get the linear combination of numbers $\pi, \Delta_1, \Delta_2, \Delta_3, \Delta_4$ with the integer coefficients $k, \{-n_i\}$. It follows from the rational independence that

$$n_1 = n_2 = n_3 = n_4,$$

and we can set all these numbers equal to 0, because the indices are defined up to an additive constant.

Then $k = 0$, i.e., the broken line γ does not make any revolution on the T's surface. This contradicts to the simplicity of contour γ : γ divides the T's surface into two parts, where each part contains at least one vertex. Therefore, the indices of vertices lying at the different parts are different which contradicts the equality $n_1 = n_2 = n_3 = n_4$. The statement is proved. ■

3.5. Stage 5: "rational" polyhedra

We will prove now that the majority of so-called *"rational"* polyhedra are free of simple closed geodesics.

Definition 6. *A polyhedron is called* "rational" *if all flat angles of each its face are commensurable with* π*, i.e., each of them is of the form*

$$\alpha_i = \frac{p_i}{q_i}\pi,$$

where $p_i, q_i \in \mathbf{Z}_+ = \{1, 2, 3, ...\}$.

Statement 6. *For any combinatorial type of convex polyhedra, there is a "rational" convex polyhedron* Im *of that type which is free of simple closed geodesics. Moreover, all rational convex polyhedra in a neighbourhood of every such polyhedron* Im *possess the same property.*

Proof.

Let us choose, according to the Statement 4 a polyhedron Im with vertices' defects $\{\Delta_i\}$. We can choose the polyhedron Im in such a way that all its flat angles are commensurable with π, and, in addition, *every* sum of defects is not equal to 2π : if I is a set of all polyhedron vertices, then

$$\sum_{\emptyset \neq J = \{j\} \subset I} \Delta_j \neq 2\pi. \tag{3.7}$$

For simple closed geodesic γ, all indices of vertices with respect to γ are either $ind_\gamma v_i = 0$ or $(+1)$ (and there are non-zero indices!), and, simultaneously,

$$0 = 2\pi k - \sum_{i=1}^{I} ind_\gamma v_i \cdot \Delta_i \ .$$

We have thus the equality

$$2\pi k = \sum_{i=1}^{I} n_i \Delta_i \ ,$$

where each $n_i = 0$ or 1, and $k = 1$. Denoting $J = \{j | \ n_j = 1\} \subset I$ we get a contradiction with inequality (10), which proves the first part of 6. The second part of the statement also follows from (10). ■

Note. Actually, we have just proved the equality

$$\sum_i ind_L v_i \cdot \Delta_i + 2\pi k = 0 \ (= \delta_L \text{ for the geodesic } L)$$

implies that all vertices' indices $ind_L v_i$ are the same – which is a contradiction.

3.6. Stage 6: generic polyhedra

Statement 7. *Let* Im *be an "irrational" polyhedron, i.e., for each proper subset $I \subset \{1, 2, ..., V\}$ the defects of all vertices v_i, $i \in I$, are rationally independent with the number π:*

$$(\sum_{i \in I} n_i \Delta_i + n \cdot \pi = 0) \Longrightarrow (n_i = n = 0, \ \forall i).$$

Then the polyhedron Im *is free of simple closed geodesics.*

Proof.

We will use the Gauss-Bonnet theorem for polygons.

Namely, let $Q = A_1 A_2 ... A_n$ be a given polygon whose boundary $\partial Q = L$ is a clockwise oriented piecewise smooth curve. Let a Riemannien metric g_{ij} be given on this polygon. Also let $K(v)$ be the (non-negative) gaussian curvature of each given inner point $v \in Q$. Let us denote the unit vector $\dfrac{\vec{A_i A_{i+1}}}{\mid \vec{A_i A_{i+1}} \mid}$ by $\vec{e}_{i+1}$ (we suppose that $A_{n+1} \equiv A_1$). Denote by φ_i the angle which the unit vector $\vec{e_i}$ should be turned for coinciding with the unit vector $\vec{e}_{i+1}$ (Fig.9), and by $d\sigma$ an area element in Q.

The Gauss-Bonnet Theorem for polygons:

$$\sum_{i=1}^{n} \varphi_i + \int_{v \in Q} K(v) d\sigma = 2\pi.$$

If the curvature $K(v) = 0$ everywhere except the finite number of points ("vertices") $v_1, ..., v_p$, then each curvature $K(v_i)$, $i = 1, ..., p$, is the *defect of vertex* v_i, and

$$K(v_i) = 2\pi - \sum_{\{\alpha_j\} \in v_j} \alpha_j = \Delta_{v_i}.$$

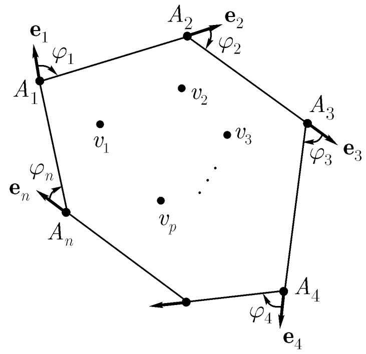

Fig. 9

Here, $\{\alpha_j\}$ are the flat angles around the vertex v_i. In this case,

$$\sum_{L} \varphi_i + \sum_{i=1}^{p} \Delta_{v_i} = 2\pi,$$

or

$$\delta_L + \sum_{i=1}^{p} \Delta_{v_i} = 2\pi.$$

If $L = \gamma$ is an arbitrary, not obligatory closed, geodesic, then $\varphi_i = 0$ for each i; therefore,

$$\sum_{i=1}^{p} \Delta_{v_i} = 2\pi. \tag{3.8}$$

Now we will use the Gauss-Bonnet theorem for polygons to simple closed geodesic γ. Cut the polyhedron Im into two parts along the broken line γ and take that one from these which contains at least one vertex; denote the number of the vertices by p. We have then the (3.8).

However, we supposed that

$$\sum_{i \in I} n_i \cdot \Delta_i \neq 2\pi k$$

for each n_i, k with respect to *every proper* subset I of the set of polyhedron's vertices, and, despite this we have found the subset consisting of p vertices, where all the Δ_{v_i} appear in the linear combination $\sum n_i \Delta_i = 2\pi k$ with coefficients $n_i = 1$ $(\forall i)$ and $k = 1$.

It gives us the seeked contradiction and proves the absence of simple closed geodesic γ on the Im's surface. ■

3.7. Stage 7: proof of Theorems 1, 2

The inequalities

$$\sum_{J=\{j\}\subset I} \Delta_j \neq 2\pi$$

define an open set of full measure in the space $\mathcal{P} \subset \mathbf{R}^{2E}$ (E is the number of edges of a given polyhedron Im in $\mathbf{R}^3$). Thus both theorems, **1** and **2**, are proved. ■

Note. Actually, we have gotten more strong result: in configuration space $\mathcal{P}$, the set of all polyhedra containing simple closed geodesics belongs to a finite union of planes of the form (11) in $\mathbf{R}^{2E}$.

4. Stability of Closed Geodesics on the Polyhedron Surface

4.1. Stable geodesics

Let Im be a convex polyhedron and γ be a closed *non-simple* (i.e., having self-intersections) geodesic on its surface; let its links (segments) belong to the faces $\Gamma_{j_0}, \Gamma_{j_1}, ..., \Gamma_{j_N}$ of the polyhedron. For simplicity of designation, we reenumerate at once these faces in the form $\Gamma_0, ..., \Gamma_N$.

Consider the plane Π of the face Γ_0. Suppose Im is in such a position that plane Π is horizontal. Roll Im on the plane Π to straighten γ into a line k on Π (see the end of the Subsection 2.1). Let us call $i_1, ..., i_N$ all the edges of Im through which we rolled the polyhedron (it is clear that $i_1 \in \Gamma_0 \cap \Gamma_1, ..., i_N \in \Gamma_{N-1} \cap \Gamma_N$).

The geodesic γ is a segment in the "corridor" $\Gamma_0\Gamma_1\Gamma_2...\Gamma_{N-1}$ thus obtained. Since γ is closed, there is an N such that Γ_N is obtained from Γ_0, and also the corresponding polyhedron Im_N from polyhedron $\mathrm{Im}_0 = \mathrm{Im}$, by parallel translation $T_{\vec{a}}$ by a vector $\vec{a}$:

$$\Gamma_N = T_{\vec{a}}\Gamma_0; \quad \mathrm{Im}_N = T_{\vec{a}}\,\mathrm{Im}_0\,. \tag{4.1}$$

Moreover, a point-like particle starting at a point A on the "horizontal" side of Γ_0 falls back to the same point A on the "horizontal" side of Γ_N (Fig. 10). Thus, every straightened closed geodesic γ joins "identical" points on the bases of the parallel polygons Γ_0 and Γ_N and lies in the corridor formed by the successive rolling of Im in those edges of Im that intersect the straightened geodesic. Here, if the length of γ (its number of links) is *even*, then this length equals N; if it is *odd*, then N is chosen to be twice the length of γ. In each case N is even.

Now, let us introduce a notion of stable closed geodesic.

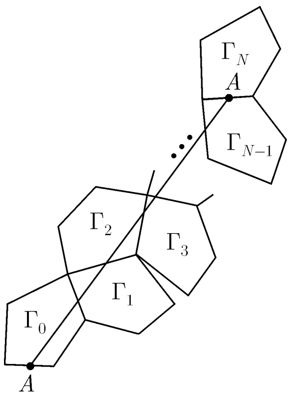

Fig. 10

Definition 7. A closed geodesic γ on Im's surface is called *stable* if it is not destroyed by any small deformation of Im. In other words, by an arbitrarily small change of Im to form a new polyhedron Im$'$ there is a closed geodesic γ' with the same number of links as γ, and each its link is close to the corresponding link of γ.

Note. The longest possible size of the deformation of Im depends on γ.

It follows from the above reasoning that if γ is a stable closed geodesic, then, by deforming Im a little bit to Im$'$ and γ to close geodesic γ' on Im$'$, and by performing N rollings in the same edges $i_1, ..., i_N$ we obtain a corridor $\Gamma_0'\Gamma_1'...\Gamma_N'$ in which there lies a straightened geodesic γ' close to the straightened geodesic γ.

Moreover, $\Gamma'_N = T_{\vec{b}}\Gamma'_0$ and $\mathrm{Im}'_N = T_{\vec{b}}\mathrm{Im}$, where $\vec{b}$ is some vector close to the vector $\vec{a}$ (Fig.11). This means that the polyhedra $\mathrm{Im}_0 = \mathrm{Im}$ and Im_N are parallel to each other precisely because of *a unique* relation between the variations $\delta\Delta_1, ..., \delta\Delta_V$ of the defects of Im: $\delta\Delta_1 + ... + \delta\Delta_V \equiv 0$ (recall that there is *the only* relation between the defects themselves: $\Delta_1 + ... + \Delta_V \equiv 4\pi$).

In other words, the stability of γ is determined by a property of the sequence of the sides $i_1, ..., i_N$ with respect to which we rolled Im. The sequence $W = i_1 i_2 ... i_n$ is called the *code-word* and it determines both the geodesic γ and the corridor $\Gamma_0, \Gamma_1, ..., \Gamma_N$.

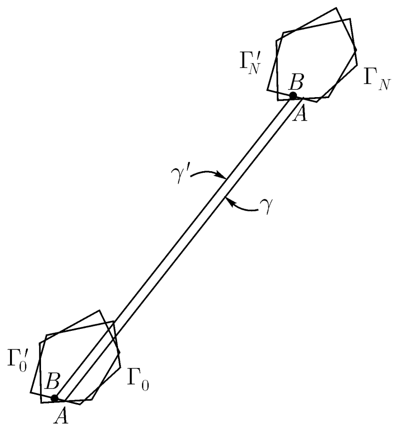

Fig. 11

4.2. Geodesics on a generic polyhedron

Let Im be *a generic* polyhedron. Recall that for generic polyhedron, the numbers in any set $\{\pi, \Delta_{j_1}, ..., \Delta_{j_k}\}$, where $\{\Delta_{j_1}, ..., \Delta_{j_k}\}$ is a proper subset of the set of vertices defects of Im, are rationally independent. The following theorem holds.

Theorem 3. *Every closed geodesic γ in a generic polyhedron* Im*'s surface is stable.*

Proof. According to the Gauss-Bonnet theorem for polyhedra (see Statement 3),

$$\sum_{v_i \in \mathrm{Im}} \Delta_{v_i} \cdot ind_\gamma v_i = 2\pi k,\ k \in \mathbf{Z}.$$

From here, we get using the genericity of polyhedron Im that all vertices indices with respect to γ are the same.

Let us make the procedure of rolling Im as it was descibed in subsection **4.1**. Then we will have equalities (12), i.e., the polygon Γ_N is parallel to the polygon Γ_0 ($\Gamma_N \parallel \Gamma_0$) in the corridor $\Gamma_0 \ldots \Gamma_N$ and the straightening of γ is a segment AA inside that corridor. After a small perturbation of Im we will get a polyhedron Im' close to Im. Making the same procedure of rolling Im' we will get a new corridor of polygons $\Gamma'_0 \ldots \Gamma'_N$, where $\Gamma'_N \parallel \Gamma'_0$ since all $ind_\gamma v_i$ are the same. Because of smallness of the perturbation, one can choose the "identical" points B, B on the bases of Γ'_0 and Γ'_N for which the whole segment BB belongs to the corridor $\Gamma'_0 \ldots \Gamma'_N$ and is close to the segment AA. Then BB is a straightening of the closed geodesic γ' on Im' which close to γ on Im. Therefore, γ is stable. ■

4.3. Criteria of stability of a geodesic

Analogous reasoning to that in [14] can be used to establish the criterion of stability of a closed geodesic in the surface of a polyhedron.

Theorem 4. (The first criterion of stability). *The geodesic γ on the arbitrary polyhedron* Im *is stable* if and only if *the following two conditions hold:*

(i) *the alternating sum of the letters in the code-word W identically coincides with* 0*:*

$$i_1 - i_2 + i_3 - ... \pm i_N \equiv 0,$$

i.e., each letter $(+i_s)$ in this summation is canceled out by the same letter $(-i_s)$ (with the opposite sign);

(ii) *the corridor constructed from the code-word W contains completely the straight line segment AA in the Fig.* 10.

Using Theorem 3 we can get now the other criterion of stability of a closed geodesic.

Theorem 5. (The second criterion of stability). *The geodesic γ in the surface of an arbitrary polhedron* Im*'s is stable* if and only if *all indices of* Im*'s vertices with respect to γ are the same.*

Proof $\Downarrow$. Let γ be a stable closed geodesic on Im. Deform polyhedron Im in such a manner that the deformed polyhedron Im$'$ is generic. Because of stability of γ there is a closed geodesic γ' on Im$'$ close to γ. Since all indices $\{ind_{\gamma'} v_i\}$ are the same we get that all indices $\{ind_{\gamma} v_i\}$ are the same as well.

($\Uparrow$). Conversely, if all indices $\{ind_{\gamma} v_i\}$ are the same for a given closed geodesic γ on Im , then Theorem 3 reasonings give us $\Gamma_N \parallel \Gamma_0$ and $\Gamma'_N \parallel \Gamma'_0$ for a perturbed polyhedron Im$'$. Therefore, there is a geodesic γ' whose straightening BB in the corridor $\Gamma'_0 \ldots \Gamma'_N$ close to the γ's straightening AA in the corridor $\Gamma_0 \ldots \Gamma_N$. Hence, γ is stable. ■

Corollary 1. The condition **(i)** from the first criterion of stability of a closed geodesic is equivalent to the equality of all indices of vertices with respect to that geodesic.

Corollary 2. Every simple closed geodesic is *unstable*.

Proof. Indeed, indices of vertices in different parts about the simple closed geodesic are distinct, and Theorem 5 gives the requirement immediately. ■

(The alternative proof follows from Theorem 3: after a small perturbation of the polyhedron containing a simple closed geodesic we can get a generic polyhedron which does not contain any simple closed geodesics at all). ■

4.4. Application to billiards in polygons

Let us now consider billiards in an arbitrary polygon Q. We take a second copy Q' of Q and glue it to Q along the boundary (identifying corresponding sides). Any path γ of a billiard particle on Q may be regarded as a geodesic on the degenerate polyhedron $\mathrm{Im} = Q \cup Q'$: after a straight line segment (a link) of the path lying in Q has been reflected, the next segment lies in Q', and conversely. The defect of the i-th vertex of Im at which the angle of the polygon is α_i is equal to $\Delta_i = 2(\pi - \alpha_i)$. The path γ is *a stable periodic billiard trajectory (orbit)* if and only if it is *stable as a closed geodesic on* Im. Therefore, theorems 3 and 5 implies the following corollaries.

Theorem 6. *Every periodic billiard trajectory in a generic polygon Q is stable.*

Note. The direct proof of Theorem 6 is cited in [14].

Theorem 7. *A periodic billiard orbit γ in polygon Q is stable* if and only if *the indices of all the vertices of polyhedron* Im *with respect to γ are the same.*

If we label the sides of the n-gon Q by the letters $a_1, ..., a_n$ and write down the siquence $a_{i_1}, ..., a_{i_s}$ of sides on which the reflections of the path γ occur, then we have

$$a_{i_1} - a_{i_2} + ... \pm a_{i_s} \equiv 0.$$

The stability criterion for a billiard trajectory in a polygon obtained in [14] has thus been generalized to the case of stability of closed geodesics on polyhedra.

Let us demonstrate some examples of both stable and unstable periodic billiard trajectories considered as closed geodesics.

Examples 1. The word W corresponding to the three-links periodic billiard trajectory in a given acute triangle (the so-called *"Fagniano tarjectory"*) has the form $W = abcabc$ (see Fig. 12a). It is stable because

$$a - b + c - a + b - c \equiv 0$$

(we need to take a path with an even number of links, 6 in our case). The corresponding degenerate polyhedron Im is shown on the Fig. 12b.

Replace the vertices of Im with small circles A, B, C, we deform Im by stretching the circle B until we get Fig. 12c (it corresponds to the stereographic projection from vertex B).

We will get then a disc whose boundary corresponds to the circle B, circles A and C are inside the disc, sides a, b, c lie at the same diameter of the disc, and the 6-links geodesic on Im is depicted as a curve γ inside the disc overlapping the sircles A and C as it is shown on Fig. 12c. It is easy to see that all three indices of vertices A, B, C with respect to γ are the same (and are equal to 1). Therefore, it confirms that the considering billiard trajectory is stable, according to the theorem 6.

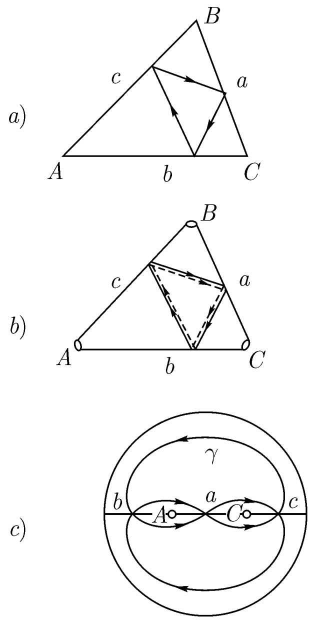

Fig. 12

2. As it has been proved in [14], all periodic trajectories in the right-angled triangles with the acute angles $(\pi/6, \pi/3)$ and $(\pi/4, \pi/4)$ are *unstable*. Therefore, for any periodic trajectory γ in any of these triangles there are two vertices with dinstinct indices with respect to γ.

This example shows that not only *simple* closed geodesics can be unstable.

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