Student Mathematics Competition
Illinois Section of the
Mathematical Association of America
North Central College, March 31, 2000

Do any four of the six problems. Put your solutions on the papers provided, beginning each problem solution on a new page. Only hand in four solutions. Entries will be graded on the basis of correctness, clarity of exposition, and elegance of solution. Enjoy the problem solving.

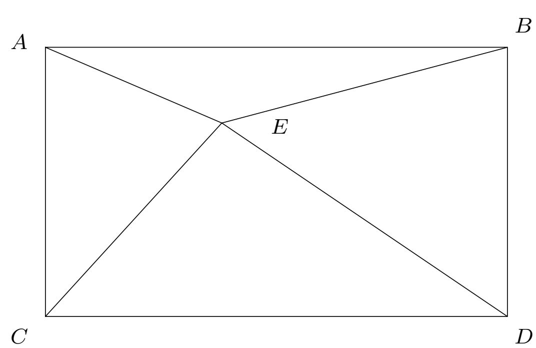
1. You roll 2000 dice one time and find that the ratio of the sum of the numbers on the top faces to the sum of the numbers of the bottom was again an integer. How many different integers are possible for the sum of the numbers on the top faces? Justify your answer

2. Prove that if $x_1, x_2, \ldots, x_n$ are positive, real numbers, then
$$\left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right)\cdots\left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right) \geq 2^n.$$

3. A snail is crawling in the garden. It starts at a certain point and crawls 15 minutes in one direction. Then it makes a 90° turn to either the left or right and crawls 15 minutes in this new direction. The snail repeats this process of crawling and turning until it returns to the starting point. Show that snail is at the starting point after a whole number of hours.

4. Evaluate $\lim\limits_{x\to 0} \dfrac{x^2 \int_0^x \sqrt{t^3+1}\; dt}{\int_0^x t^2 \cos t\; dt}$.

5. Suppose $ABCD$ is a rectangle and E is a point such that $AE = 1$, $CE = 2$ and $DE = 3$, as shown below. Find all possible lengths for BE and justify your answer(s).



6. Let n be a positive number. Show there are a finite number of solutions to the equation
$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = 1,$$
where $x_1, x_2, x_3, \ldots, x_n$ are positive integers.