# Challenges of the Week
## Solutions
## Spring Semester 2008-2009

*Challenge of the Week # 1 - January 21 to January 30: A street hustler is trying to take your money. He has four cards in front on him. The first is a Jack, the second a Queen, the third a King and the last card is an Ace. He turns over all cards. He interchanges the first and third card, then the second and third and finally the first and last cards. He repeats this process of three interchanges 19 times. What is the final location of the Queen? Justify your answer.*

**Solved by Daniel Dombowski, Hilary Cloe, Randall Gaffner, Robert Hamilton, Ryan Hoklas, Randy Kuhl, Joseph Leipert, Nicole Mulrooney, Mollie Neff, Peter Provin, Tim Russell, Fremont Schneider, Ashlee Sharp, Andrew Short, Dominique Sims, Rueben Stence, and Max Sweeney. A partial solution was provided by Hanna McHugh and a submission also came from Andrew Short. The following solution was given by Tim Russell, who also provided a computer program which solved the problem.** To make our problem easier, we can first index the cards as follows:

$$\begin{array}{cccc} \text{Jack} & \text{Queen} & \text{King} & \text{Ace} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{array}$$

Then we see that the initial placement of cards can be written simply as: $(1, 2, 3, 4)$. Now, the first interchange of cards is the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

The second interchange is then

$$\begin{pmatrix} 3 & 2 & 1 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$$

and finally, the third interchange is

$$\begin{pmatrix} 3 & 1 & 2 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

But we see that the composition of these three interchanges results in a permutation that is just a right "rotation" of the cards, with the card on the right "wrapping" around to the first position. Considering these three operations as the one just described, we can quickly see that four of these rotations will result in the identity permutation: that is, after performing the group of three permutations four times, the result will be the set of cards returned to the configuration $(1, 2, 3, 4)$.

It is then easy to see that, since the rotation has a period of four, that 16 rotations will result in the original configuration again, and the last three of the hustler's rotations will leave the cards in the configuration:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 4 & 1 & 2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

which leaves the Queen in the first position.

*Challenge of the Week # 2 - January 30 to February 6: Consider the following process:*

> *A positive integer $n$ is chosen. Then the product $n(n + 1)$ is computed and two digits are appended to the end of the product. The resulting number is the square of an integer.*

*Show that it is always possible to complete the process above or give a positive integer $n$ for which the process cannot be completed.*

**There were 4 correct solutions submitted to this week's challenge. The two winners of the drawing for $10 each were Jon Harter and Max Sweeney. Correct solutions were also provided by Joseph Leipert and Ashlee Sharp, Submissions also came from Randall Gaffner and Cyndee Tucker. Claim:** Appending 25 to the product $n(n+1)$ always results in a perfect square. This is because appending 25 to a number corresponds to multiplying the number by 100 and then adding 25, and then noting that $100n(n+1)+25 = 100n^2+100n+25 = (10n+5)^2$. Thus it is always possible for the process to be completed.

*Challenge of the Week # 3 - February 6 to February 13:*

*This is a very challenging challenge*

*Two distinct positive integers are chosen. The square of the smaller integer is computed. The larger of the two original integers is replaced with the absolute difference of the two original integers. The process continues with the absolute value and the original smaller integer. From these, the square of the smaller integer is computed and the larger integer is again replaced with the absolute value of the difference. This process is continued until the difference is zero. What is the sum of the squares that have been computed? Justify your answer.*

**There were 2 correct solutions submitted to this week's challenge. The two winners were Hilary Cloe and Max Sweeney. Max Sweeney's solution is below. Partial solutions were given by Joseph Leipert, Ashlee Sharp, Dominique Sims, and Cydnee Tucker. An entry was also submitted by Jaimee Phegley.** Start with a rectangle whose sides have lengths $a$ and $b$, with $a < b$. Chop off a square with side length $a$ from one end of this rectangle, leaving a smaller rectangle with sides $a$ and $b-a$. Then repeat this process with the smaller rectangle. The sum of the squares that we are computing is the sum of the areas of the squares that we chop off of the rectangles. When the process ends, the difference between the numbers is zero so the rectangle is a square. Thus we have partitioned the original rectangle into squares which means that the sum of the squares equals the area of the original rectangle which is $ab$.

*Challenge of the Week # 4 - February 20 to February 27: Several real numbers (not necessarily all different) are chosen. The sum of these numbers is 10. Is it possible that the sum of the squares of these numbers is less than one-millionth? Justify your answer.*

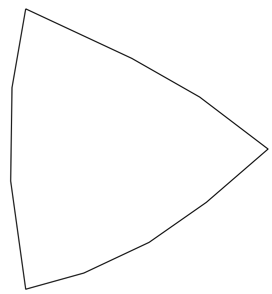
**There were 5 correct solutions submitted to this week's challenge. The two winners that won the drawing for $10 were Nicole Mulrooney and Ashlee Sharp. Jared Evenson, Joseph Leipert, and Tim Russell gave correct solutions. Max Sweeney and Cyndee Tucker submitted partial solutions.** Suppose that there are $n$ numbers that are all equal to $x$. Since the sum is 10 we have that $nx = 10$ and $x = 10/n$. Then the sum of the squares is $n(\frac{10}{n})^2 = \frac{100}{n}$. We want this sum to be less than $\frac{1}{10^6}$. Solving $\frac{100}{n} < \frac{1}{10^6}$ for $n$ gives $n > 100,000,000$. Thus it can be done but it requires more than 100 million numbers. I would call this more than the several numbers indicated in the problem.

Now suppose that the $n$ numbers are not necessarily all equal.
**Claim:** The minimum sum of the squares occurs when they are all equal. As you learn in Calculus III, you can prove this by using Lagrange multipliers. Let $f(x_1, x_2, ..., x_n) = x_1^2+x_2^2+...+x_n^2$ subject to the constraint $g(x_1, x_2, ..., x_n) = x_1+x_2+...+x_n = 10$. The minimum must occur when $\nabla f = \lambda \nabla g$ or $< 2x_1, 2x_2, ..., 2x_n >= \lambda < 1, 1, ..., 1 >$. Thus the $x_i's$ all equal $\lambda/2$, so the $x_i's$ must all be equal when $f$ is minimal. Thus, the above method with $n$ equal numbers is the best method.

*Challenge of the Week # 5 - February 27 to March 6:*

1. *Prove there exists a convex 10-gon that has 3 acute interior angles.*
2. *Prove there does not exist a convex 10-gon that has 4 acute interior angles.*

**There were 6 correct solutions submitted to this week's challenge. The two winners of the drawing for $10 each were Nicole Radecki and Max Sweeney. Correct solutions also came from Joseph Leipert, Christopher McMillan, Ashlee Sharp, and Cydnee Tucker.** Part 1: The following drawing of a 10-gon in which three of the angles are slightly more than $60^\circ$ shows that it is possible to draw a 10-gon with 3 acute interior angles.



Part 2: The sum of interior angles of a regular $n$-gon is $180(n-2)$ and the sum of the angles of a convex 10-gon is $8 \cdot 180^\circ$. If a 10-gon had 4 acute interior angles, then the sum of these four interior angles would be less than $4 \cdot 90 = 2 \cdot 180^\circ$. The remaining 6 angles are less than $180^\circ$ each and thus, the sum of all ten angles is less than $8 \cdot 180^\circ$. This is a contradiction.

*Challenge of the Week # 6 - March 6 to March 13: Find the smallest positive integral value of $n$ for which the following product is an integer.*

$$\left(1\frac{1}{36}\right)\left(1\frac{1}{37}\right)\cdots\left(1\frac{1}{n}\right)$$

*Justify your answer.*

**There were 8 correct solutions submitted to this week's challenge. The two winners of the drawing for $10 each were Christopher McMillan and Nicole Mulrooney. Correct solutions also came from Hilary Cloe, Joseph Leipert, Jaimee Phegley, Peter Provin, Max Sweeney, and Bill Wolf.**

$$\left(1\frac{1}{36}\right)\left(1\frac{1}{37}\right)\left(1\frac{1}{38}\right)\cdots\left(1\frac{1}{n-1}\right)\left(1\frac{1}{n}\right) =$$

$$\left(\frac{37}{36}\right)\left(\frac{38}{37}\right)\left(\frac{39}{38}\right)\cdots\left(\frac{n}{n-1}\right)\left(\frac{n+1}{n}\right) =$$

$$\frac{n+1}{36}$$

Since $n \geq 36$, the above expression is larger than 1. Thus the smallest such $n$ is when $\frac{n+1}{36} = 2$, so $n = 71$.

*Challenge of the Week # 7 - March 23 to April 3: This is the annual April Fool's Challenge of the Week.*

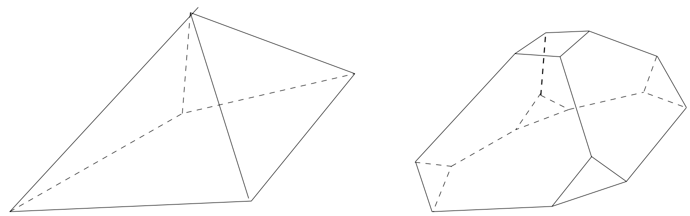
1. *A clock strikes six times in five seconds. How many times will it strike in 10 seconds? Justify your answer.*
2. *Let $\gamma$ be a closed non-intersecting curve all of whose points are 1 inch from a given point. Is it possible that $\gamma$ is longer than 1 mile? Justify your answer.*
3. *Let $\triangle ABC$ be a triangle all of whose sides are shorter than 1 inch. Can the circle which passes through $A$, $B$, and $C$ have circumference which is larger than 1 mile? Justify your answer.*

**There was one correct set of solutions for all three problems submitted to this week's challenge. The winner is Jody Alford. Quite nice solutions were also submitted by Joseph Leipert, Christopher McMillan, Nicole Mulrooney, Peter Provin, Fremont Schneider, Ashlee Sharp, Max Sweeney, and Cydnee Tucker.**

1. Any answer, five or larger, could be justified. The clock could strike at a different rate during the second five seconds than the first 10 seconds. Probably the most reasonable assumption is that the clock strikes at a uniform rate and the timing starts when the clock strikes the first time. With this interpretation, the clock strikes 11 times — once at the beginning of the time period, once at the end, and nine times in between. It is also reasonable to argue that the only thing that we know is that clocks don't strike more than 12 times in a short period of time so the answer must be less than or equal to 12.

2. Yes. Consider a path on the surface of a ball that is 1 inch in radius. One specific way to do it is to start at an equator of the ball and spiral around the ball very close to the equator, but slowly moving away from it. Since the path has no width we have plenty of room to make any number of revolutions that we wish. If we spiral around until we have more than 1/2 mile in length and then turn around and spiral right next to the original path back to the start we get a closed non-intersecting path longer than 1 mile.
3. Yes. Take a large circle of circumference larger than 1 mile and let $A$, $B$, and $C$ be three points on the circumference which are within 1 inch of each other.

*Challenge of the Week # 8 - April 3 to April 10: A small pyramid is chopped off of each vertex of a convex polyhedron $\mathcal{P}$. The result is a truncated polyhedron $\mathcal{T}$. This process is illustrated below when $\mathcal{P}$ is a square pyramid.*



*Suppose the original polyhedron $\mathcal{P}$ had 100 edges.*

- *How many edges are there in $\mathcal{T}$?*
- *How many vertices are there in $\mathcal{T}$?*

*Justify your answers.*

**There was one completely correct solution submitted for this week's challenge. The winner is Christopher McMillan. Very nice nearly complete solutions were also submitted by Joseph Leipert, Allen Obendorf, and Cydnee Tucker.** Let $V$ be the number of vertices of $\mathcal{T}$ and $E$ be the number of edges of $\mathcal{T}$. In the truncated polyhedron there are 4 'new' edges which meet each 'old' edge. But each of the 'new' edges meets two 'old' edges. Hence the number of 'new' edges is twice the number of 'old' edges. Since there are 100 edges in $\mathcal{P}$, there 200 'new' edges. Therefore, $E = 300$.
To count the number of vertices of $\mathcal{T}$, go to each vertex of $\mathcal{T}$ and place a mark on each each edge which meets that vertex. Since there are precisely three edges (two 'new' and two 'old') at each vertex of $\mathcal{T}$, there are $3V$ marks. But, since each edge of a polyhedron meets two vertices, there are 600 marks. Hence $3V = 600$ or $V = 200$.

*Challenge of the Week # 9 - April 10 to April 17: Let $A$ be the number which consists of 2009 digits all equal to 2. Let $B$ be the number which consists of 2009 digits all equal to 3. Compute $A + B^2$ and explain your computation.*

**Four completely correct solutions were submitted by Christopher McMillan, Ashlee Sharp, Dominique Sims, and Max Sweeney. The drawing for the prize money was won by Ashlee Sharp and Dominique Sims. Other nearly complete solutions were submitted by Randall Gaffner, Timothy Going, Josheph Leipert, Mollie Neff, and Cydnee Tucker.** This number equals a number consisting of 4018 digits all equal to 1. If $C$ is the number consisting of 2009 digits all equal to 1, then $A = 2 \cdot C$ and $B^2 = 9 \cdot C^2$. Hence,

$$A + B^2 = 2 \cdot C + 9 \cdot C^2 = C(2 + 9 \cdot C).$$

Now $9 \cdot C$ is the number consisting of 2009 nines so $2 + 9 \cdot C$ has 2010 digits, the first and last being 1 and the other 2008 are all 0. It is now easy to see that when this number is multiplied by $C$, the result has 4018 digits all of which are 1.

*Challenge of the Week # 10 - April 17 to April 24: Two nonzero real numbers are chosen. Which is larger — the cube of the sum of the squares of the the two numbers or the square of the sum of the cubes of the two numbers? Or, is it that in some cases the cube of the sum of the squares is larger and in other cases the square of the sum of the cubes is larger? Justify your answer.*

**The winner is Joseph Leipert. Nearly complete solutions were also submitted by Hilary Cloe, Jaimee Phegley, Peter Provin, Ashlee Sharp, and Cydnee Tucker.** The cube of the sum of the squares is always larger. Let $x$ and $y$ be the two numbers. The cube of the sum of the squares is $(x^2+y^2)^3$ while the square of the sum of the cubes is $(x^3+y^3)^2$. Subtracting the second of these from the first and simplifying gives

$$\begin{aligned}(x^2+y^2)^3 &- (x^3+y^3)^2\\ &= (x^6+3x^4y^2+3x^2y^4+y^6)-(x^6+2x^3y^3+y^6)\\ &= (x^2y^2)(3x^2+3y^2-2xy)\\ &= (x^2y^2)(2x^2+2y^2+(x^2-2xy+y^2))\\ &= (x^2y^2)(2x^2+2y^2+(x-y)^2).\end{aligned}$$

Since $x^2y^2$ and $2x^2+2y^2+(x-y)^2$ are positive. it follows that $(x^2+y^2)^3 > (x^2+y^2)^3$, as claimed.