Challenges of the Week
Solutions
Spring Semester 2007-2008

Challenge of the Week # 1 - January 11 to January 18:

(a) *Two people are playing a game. The players take turns placing coins on a large rectangular table. The coins may not overlap and they must lie entirely on the table. The player who places the last coin on the table wins. Who wins? And, if one of the players can always win with the appropriate strategy, what is the winning strategy?*

(b) *After this game, the two people go to a different game. They take turns placing a circular magnets on a very large metal, spherical ball. With the same rules for winning as in the previous game, who wins? What is the winning strategy, if there is one?*

The following solutions where submitted by Matt Niemerg

(a) Player 1 has the winning strategy. Let a coordinate system be defined on the table so that the origin is the center and the axes run parallel to the edges of the rectangular table. Player 1 places a coin at the origin. Suppose Player 2 places a coin centered at (x, y), Player 1's strategy is to place a coin at $(-x, -y)$. Since Player 1 can always mimic a symmetrical move, Player 2 will be the first player not able to place a coin on the table. Hence, Player 1 wins.

(b) Player 2 wins. The following is Player 2's strategy: Place a coin directly opposite on the sphere from Player 1's coin. If Player 1 is able to place a coin, Player 2 is able as well.

Challenge of the Week # 2 - January 18 to January 25: The two people from last week's challenge are still playing games.

(a) *One coin is placed in the middle of a large rectangular table. Now the two people take turns placing coins on the table. The coins do not overlap and each lies entirely on the table. The player who places the last coin on the table wins. Can one of the players always win? If so, what is her/his winning strategy.*

(b) *They change the rules of the game again. This game is the same as the last game, except that, before play begins, a coin is placed on the table, but **not** in the middle of the table. Who wins this game? If one player can always win, what is the winning strategy?*

There was a submission from Lindsay Harper, but no correct solution was received for these challenges.

(a) The second player can always win this game. Let a coordinate system be introduced so that the origin is at the center and the axes run parallel to the edges of the table. Because of the coin in the center of the table, every possible position at which a coin can be placed is centrally symmetric to another point at which a coin can be placed. If the first player places a coin at, say (x, y), the second player can place a coin at $(-x, -y)$ and leave the coins on the table in a symmetrical position. Thus, every move by the first player destroys the symmetry, but the second player can always restore the symmetry on his/her next move. Thus the second player will win, since she/he will always be able to move.

(b) There probably isn't a strategy for this game since the moves depend so much upon the size of the table and the size of the coin. Because the initial position is not symmetric, the winner depends upon how many coins will fit on the table. If the coins on the table are centrally symmetric, with the origin occupied, and if there is a position on the table for one coin, then there is a centrally symmetric position at which a second coin can be placed. Further, after the placement of this coin, the coins on the table are again centrally symmetric. To see the difficulty involved if there is no central symmetry, take a square whose edges are slightly more than $1\frac{3}{4}$". Three quarters will fit in the square, but it also possible to place two quarters in the square so that a third quarter cannot be placed in the square.

Challenge of the Week # 3 - January 25 to February 1: Find all three digit numbers, n, which have the following property: The three rightmost digits of its square, n^2, form the same integer n. Justify your answer.

JuLee Simpson submitted a correct solution to this week's challenge. Lindsay Harper also submitted a paper. The numbers are 376 and 625.
Suppose n is a number with the given properties. Since the last three digits of n^2 and the last three digits of n are the same, the last three digits of $n^2 - n$ are zero. That is, 1000 divides $n^2 - n = (n-1)n$ which is the product of two consecutive integers. Since 5^3 divides $n(n-1)$ and 5 cannot divide both n and $n-1$, either n or $n-1$ is a multiple of $5^3 = 125$. Since 2^3 divides $n(n-1)$, a similar argument show that 8 divides n or $n-1$. Since no three digit number is a multiple of both 8 and 125, it follows that one of n and $n-1$ is an odd multiple of 125 while the other is a multiple of 8. There are several cases to consider:

$$124, 125; 125, 126; 374, 375; 375, 376; 624, 625; 625, 626.$$

The only numbers from among 124, 126, 374, 376, 624, 626 which are multiples of 8 are 376 and 624. Hence, $n = 376$ or $n = 625$.

Challenge of the Week # 4 - February 1 to February 8: Let a, b, and c be three unknown real numbers which are not integers. Let

$$n = a - b + 2008, m = b - c + 2008, k = c - a + 2008.$$

Suppose that m, n and k are three consecutive integers. Find the three integers. Justify your answer.

Correct solutions were submitted by Jody Alfred, Philip Boehl, Dan Campbell, and Ben Thompson The three consecutive integers are 2007, 2008, and 2009.
Adding the three equations together gives $n + m + k = 3 \cdot 2008$. Since n, m, k are consecutive integers, they are, in some order $a-1$, a, $a+1$ for some $a \in \mathbb{Z}$. Substituting these values into the previous equation gives $3a = 3 \cdot 2008$. The result follows immediately.

Challenge of the Week # 5 - February 8 to February 22: Since there is no class on February 15, this will be a "Challenge of the Two Weeks".
The sum of two positive integers is 2008. Suppose that if you divide the larger integer by the smaller one, you get a positive remainder r. What is the largest possible value of r? Justify your answer.

There were two exellent solutions to this week's challenge. Matt Neimerg submitted a solution similar to the one given below and Tim Russell submitted a C++ program which solved the problem.

Let the two numbers be x and y where $x > y$ and $x + y = 2008$. By long division, there is a quotient, q, and a remainder, r, with $x = qy + r$, where $0 \leq r < y$.

If $q \geq 2$, then $x \geq 2y + r$. Adding y to both sides of this inequality gives $2008 = x + y \geq 3y + r > 4r$. Hence $r < 502$. If $q = 1$, then $x = y + r$ and $2008 = x + y = 2y + r \geq 2(r+1) + r$. Thus, $2006 \geq 3r$ and $r \leq 668$. Since $x > y$, q must be 1 or larger. Therefore, the largest possible remainder is at most 668.

The maximum can be attained since $2008 = 1338 + 670$ and $1338 = 1 \cdot 670 + 668$.

Challenge of the Week # 6 - February 22 to February 29: A person is in the middle of round pool which is 200 feet in diameter. A very vicious dog is running around the outside of the pool, but will not enter the pool because it is afraid of the water. The dog can run around half of the circumference of the pool in the same time that it takes for the person to swim half of the diameter of the pool. In order to escape, the person must reach the edge of the pool before the dog gets there. Can the person escape, or not? Justify your answer.

There were no correct solutions given to this week's challenge. However, Julee Simpson and Peter Provin submitted attempts. The person swims out one-quarter of the radius of the pool. The person then swims around the circle, centered at the center of the pool, until he/she is directly opposite the dog, at which time the person swims to the closest point on the edge of the pool.

To see that this method will work, let a be the speed of the person, b be the speed of the dog, and r the radius of the pool. The time required for the person to swim the radius of the pool is r/a The time required for the dog to run around half of the circumference of the pool is $r\pi/b$. . Since these are equal $a = b/\pi$.

The time required for the person to swim around a circle of radius $r/4$ is

$$\frac{2\pi(\frac{r}{4})}{a} = \frac{r\pi}{2a} = \frac{r\pi^2}{2b}.$$

The time required for the dog to run around the pool is $\frac{2r\pi}{b}$. Since $\pi < 4$,

$$\frac{r\pi^2}{2b} < \frac{4r\pi}{2b} = \frac{2r\pi}{b}.$$

Therefore, the person can swim around the small circle faster than the dog can run around the pool. This means that the person can always swim around the circle to obtain the situation where he/she, the center of the pool and the dog are in a straight line with the center of the circle in the middle. At this time, the distance from the person to the closest point on the radius of the pool is less than the radius of the pool while the distance that the dog must run to get to that point is one-half of the circumference of the pool. Therefore, the person can escape.

Challenge of the Week # 7 - February 29 to March 7: Does there exist a positive integer n, whose square equals

$$7^{198} + (2^{200} \cdot 7^{99}) + 2^{398}?$$

If yes, is n a prime number of a composite number? Justify your answer

This week's challenge was solved by Matt Niemerg. A submission was also received from Marcus Waller. By factoring

$$7^{198} + (2^{200} \cdot 7^{99}) + 2^{398} = (7^{99} + 2^{199})^2$$

we see that the number is a perfect square.

It remains to show $7^{99} + 2^{199}$ is not a prime. Now,

$$7^{99} - 1 = (7 - 1)(7^{98} + 7^{96} + \cdots + 7 + 1)$$

while

$$2^{199} + 1 = (2 + 1)(2^{198} - 2^{196} + \cdots - 2 + 1).$$

Thus, $7^{99} - 1$ and $2^{199} + 1$ are multiples of 3. Therefore, their sum, $7^{99} + 2^{199}$, is a multiple of 3 and thus it is not a prime.

Challenge of the Week # 8 - March 21 to March 28: A cubical box contains some marbles. Each of the six faces of the box is labelled with a different number chosen from 1, 2, 3, 4, 5, 6. Also, each face contains the following statement:

The number of marbles in this box is greater than $10 + 2k$,
where k is the label on this face.

If half of the statements are true and half are false, how many marbles are in the box? Justify your answer.

Robert Johnson and Matt Niemerg submitted complete solutions to this week's challenge. Partial solutions were provided by Mollie Neff and JuLee Simpson. The statements are "The number of marbles in this box is greater than 12", "The number of marbles in this box is greater than 14", . . . , "The number of marbles in this box is greater than 22". Consider the statements in the order indicated. If one of the statements is true, then all of the statements before that statement are also true. Since three statements are true, the first three, in the given order, are true, while the last three are false. In particular, it is true that the number of marbles is greater than 16, but false that the number of marbles is greater than 18. Hence, there could be either 17 or 18 marbles in the box.

Challenge of the Week # 9 - March 28 to April 4:

This is the annual April Fool's edition of the Challenge of the Week.

1. *One person is standing on the balcony. On the ground immediately below, another person is standing. The two people shout "Hey" at exactly the same time. Who hears the other person's shout first, the person on the ground or the person on the balcony?*
2. *Find the two leftmost digits of the following number:*

$$1^1 + 2^2 + 3^3 + \cdots + 999^{999} + 1000^{1000}.$$

3. *Put one mathematical sign between the digits* 5 *and* 6 *to get a number between 5 and 6. (Note:* $5-6$, $5\sqrt{6}$, *and* $5/6$ *will not work.)*
4. *Are there two different sequences of 9 consecutive integers such that the product of the integers in one sequence equals the product of the integers in the other sequence? How many pairs of such sequences are there?*

In each case, justify your answer.

Correct solutions to all parts of this week's challenge were given by Robert Johnson.

1. The person on the ground hears the sound first because the distance between the bottom person's ears and the top person's mouth is less than the distance from the bottom person's mouth and the top person's ears.
2. The number 1000^{1000} is much larger than any of the other numbers. In fact,

$$1 + 2^2 + \ldots + 999^{999} < (1000) + (1000)^2 + \ldots + (1000)^{999} = \frac{(1000)^{1000} - 1}{999} < \frac{(1000)^{1000}}{100}.$$

   Therefore, the left two digits of the sum are the same as the left two digits of $(1000)^{1000}$. These are the digits 1 and 0.
3. 5.6 is between 5 and 6.
4. Let f be the function $f(a) = (a)(a+1)(a+2)\cdots(a+8)$.

   Claim: Suppose a and b are different integers each of which is either positive or less than -9. The product of the 9 consecutive integers beginning with a is different than the product of the nine consecutive integers beginning with b.

   Proof of Claim: Suppose for such a pair of integers that $f(a) = f(b)$ with $a < b$. Now, $f(a) < 0$ if and only if a is negative. Hence, a and b are either both negative or both positive. If $0 < a < b$, then $f(a) < f(b)$. If $a < b \leq -9$, $f(a) > f(b)$. This establishes the claim.

   Since $f(-8) = f(-7) = f(-6) = f(-5) = f(-4) = f(-3) = f(-2) = f(-1) = f(0)$, there are 9 sequences of nine consecutive integers whose product is zero and thus there are $\binom{9}{2} = 36$ pairs of sequences.

Challenge of the Week # 10 - April 4 to April 11: Two trains run on parallel tracks, one train is 1200 feet long and the other train is 1500 feet long. Each train runs at a constant speed. When they are going in the same direction, it takes 75 seconds for one train to completely overtake the other. When they are going in opposite directions, it is only 15 seconds between time that the heads of the two trains are even to when the ends of the trains are even. Determine the speed of the longer train. Note: There could be two answers!

There were no complete solutions to this week's Challenge, although Neil Wahlgren submitted a partial solution. Let s be the speed (in ft/sec) of the longer train and t be the speed of the shorter one.

Suppose the two trains are going in opposite directions and consider the moment at which the heads of the trains are even. In the next 15 seconds, the sum of the distance travelled by the longer train and the distance traveled by the shorter train will be the sum of the lengths of the two trains. That is,

$$15s + 15t = 2700$$

or

$$s + t = 180$$

Now, suppose the two trains are going in the same direction and consider the moment at which the tail of the slower train is even with the head of the faster train. In the next 75 seconds, the faster train will go a distance equal to the sum of the lengths of the trains inorder to completely overtake the slower train. That is,

$$\begin{aligned} 75s - 75t &= 2700 && \text{if the longer train is faster} \\ 75t - 75s &= 2700 && \text{if the shorter train is faster} \end{aligned}$$

or

$$\begin{aligned} s - t &= 36 && \text{if the longer train is faster} \\ t - s &= 36 && \text{if the shorter train is faster} \end{aligned}$$

If the longer train is faster, its speed is $s = \frac{1}{2}((s+t) + (s-t)) = 108$ ft/sec and the speed of the smaller train is 72 ft/sec. If the shorter train is faster, the speeds are reversed.