Challenges of the Week
Solutions
Spring Semester 2000-2001

Challenge of the Week # 1 - January 19 to January 26: You have two containers, one labelled A, containing 100 liters of water, and another labelled B, which is initially empty. On the first step, you take one-half of the water in A and transfer it to B. On the second step, you take one-third of the water in B and transfer it to A. On the third step, you take one-fourth of the water in A and transfer it to B. On the fourth step, you take one-fifth of the water in B and transfer it to A. You continue in this manner. How much water is in A after 2001 steps? Justify your answer.

Solved by Stephanie Apgar, Eric Bunow, Peter Brusky, Donnie Cantrell, Will Cannon, Brandon Cox, Kevin Christian, Laura Duxbury, Matt Frankie, Miranda Heuerman, Holly Hussong, Jessica McQueen, Jameel Mirza, Lexie Nuckolls, Cassie Schultz, Symon Shmukler, Erica Shifflet, Melissa Wade We show that after each odd step half of the liquid is in A. Clearly, after the first step, half of the liquid is in A. Suppose after the $2k-1$-st step that half of the liquid is in A. Now consider freezing both containers and "cutting" the contents of each into $2k+1$ equal parts. Since both containers contain half of the liquid all frozen parts will be of the same size.

On the $2k$-th step, we are to divide the liquid in B into $2k+1$ equal parts parts and transfer one part to A. In terms of "frozen parts", this means we are to transfer one frozen part to A. On the $2k+1$-st step, we are to divide the liquid in A into $2k+2$ equal parts and transfer one part to B. But there are $2k+2$ frozen parts in A, each of equal size. So after the transfer, exactly half of the frozen parts will be in A and half will be in B.

Challenge of the Week # 2 - January 26 to February 2: The following long division calculation is obviously incorrect. The pattern is correct and there is no remainder. However, all of the figures are off by 1, that is they are 1 more or 1 less than the correct figures. Find the correct figures.

$$\begin{array}{cc|ccc} & & & 3 & 4 \\ \hline 1 & 4 & 6 & 1 & 8 \\ & & 5 & 5 & \\ \hline & & & 7 & 8 \\ & & & 5 & 8 \\ \hline \end{array}$$

Justify your answer.

Solved by Peter Brusky, David Burke, Will Cannon, Brandon Cox, Carrie Doll, Laura Duxbury, Ann Foreman, Donald Ford, Holly Hussong, Megan Tobin, Roberta See, Erica Shifflet, Symon Shmukler From the last two lines of the division, we can see that the correct first digit for the number in the last line is 6 and the second digit is either 7 or 9. But the last line is a number which is obtained by multiplying a one digit number by a two digit number. Since 67 is a prime number and $69 = 3 \cdot 23$, the number in the last line is 69, the divisor is 23 and last digit of the quotient is 3. The first digit of the quotient is either 2 or 4 and thus the third line (containing 55 in the original problem) is either 46 or 92. Only 46 fulfills the conditions of the problem. The rest of the digits can now be filled in to get the correct division:

$$\begin{array}{cc|ccc} & & & 2 & 3 \\ \hline 2 & 3 & 5 & 2 & 9 \\ & & 4 & 6 & \\ \hline & & & 6 & 9 \\ & & & 6 & 9 \\ \hline \end{array}$$

Challenge of the Week # 3 - February 2 to February 9: In the figure below, each of the letters A through I stands for a different digit (1 through 9). Determine all of the ways in which this can be done so the sum of the digits in each of the two rows and each of the two columns is 13.

$$\begin{array}{ccccc} A & B & C & & \\ & & D & & \\ & & E & F & G \\ & & & & H \\ & & & & I \end{array}$$

Justify your answer.

Some solutions obtained by Will Cannon, Anne Carlson, Joshua Cottril, Brandon Cox, Annie Corl, Jessie DeLio, Steve Drake, Laura Duxbury, Jonathan Eilers, Linsey Howe, Holly Hussong, Kristin Laribee, Amber Mauk, Amanda Meehan, Barry Meyer, Stephanie Michaelis, Jeremy Pekas, Erica Shifflet Symon Shmukler, Lori Stutzman. Mitch Upton, Beth Vidic and Katie Virtue Adding up the numbers in the two rows and the two columns gives

$$A + B + C + C + D + E + E + F + G + G + H + I = 4 \cdot 13 = 52$$

But the left hand side of this sum is $(A + B + \cdots + I) + (C + E + G)$, which equals $45 + C + D + E$. Hence $52 = 45 + C + E + G$. It follows that $\{C, E, G\} = \{1, 2, 4\}$. It is not possible to combine 1 and 2 with another digit and obtain a sum of 13. Therefore the middle digit, E, is 4. The possiblities can now be quickly found.

9	3	1			9	3	1			3	9	1			3	9	1		
		8					8					8					8		
		4	7	2			4	7	2			4	7	2			4	7	2
				5					6					5					6
				6					5					6					5
6	5	2			6	5	2			5	6	2			5	6	2		
		7					7					7					7		
		4	8	1			4	8	1			4	8	1			4	8	1
				3					9					3					9
				9					3					9					3

Challenge of the Week # 4 - February 9 to February 19: You have 100 sticks, each of a slightly different length. You want to know if each triple of sticks can be made into a triangle. (Recall that three lengths can be used as the lengths of the sides of a triangle if and only if the sum of any two of the lengths is greater than the remaining length.) One way to determine if each triple can be made into a triangle is to check all possible triples (there are 161,700 such triples). Can you determine if a triangle can be made from every triple of sticks, but check a smaller number of triples? Justify your answer. The award will go to the solution which checks the fewest number of triples.

No correct solutions were received for this week's challenge Suppose the sticks are arranged in order of increasing length and then labelled $x_1, x_2, x_3, \cdots, x_{100}$. For convenience, the label on the stick will be used to denote the length of the stick. It is possible to make a triangle from all 161,700 triples if and only if and only if it is possible to make a triangle from x_1, x_2, and x_{100}.

Obviously, if it is possible to make a triangle from every triple it is possible to make a triangle from a specific triple of sticks. Suppose it is possible to make a triangle from x_1, x_2, x_{100}. Since the sum of any two of these numbers is larger than the third, we must have $x_1 + x_2 > x_{100}$. Suppose we have any triple of sticks. We must show any two of the lengths in this triple is larger than the remaining length.) Pick two of the lengths of this new triple, label the shorter length x_r and the longer length x_s. Let x_t be the remaining length. Since x_1 is the smallest of all 100 length, $x_1 \leq x_r$. Since x_2 is the second smallest of all of the lengths, $x_2 \leq x_s$. Since x_{100} is the longest of all of the lengths $x_{100} \geq x_t$. Therefore,

$$x_t \leq x_{100} < x_1 + x_2 \leq x_r + x_s,$$

and the result is established.

Challenge of the Week # 5 - February 23 to March 2: A positive integer can be transformed to another by the following procedure: If n is the original number, first find two positive integers a and b such that $a + b = n$ and then replace n by ab.

By using this process, 19, which equals $17 + 2$, can be transformed to 34; 34, which equals $30 + 4$, can be transformed to 120; and 120, which equals $20 + 100$, can be transformed to 2000. Of course, there are several different numbers to which a given number can be transformed. For example, 8 can be transformed to 12 or 15, as well as several other numbers.

Is it possible to use the method above to change 20 to 2001 in a sequence of transformations.? Justify your answer.

Solved by Laura Duxbury, Adam Trant, and Symon Shmukler There are many ways to do this transformation, the shortest is the following:

$$20(= 17 + 3) \rightarrow 51$$

$$51(= 49 + 2) \rightarrow 98$$

$$98(= 69 + 29) \rightarrow 2001.$$

Challenge of the Week # 6 - March 2 to March 9: Four soccer teams - A, B, C, D - are to play a round robin tournament in which each team plays each other team exactly once. The following table gives some of the information regarding games played, games won, games lost, etc. after some (possibly all) of the games have been played.

Team	*Games Played*	*Won*	*Lost*	*Drawn*	*Goals For*	*Goals Against*
A	*3*				*2*	*0*
B					*2*	*1*
C				*0*	*2*	
D					*0*	*2*

It is known that in each game of the tournament, with one exception, at least one of the two teams scored exactly one goal. Who played whom, and what was the score of each games?

No correct solutions were submitted to this week's challenge. The sum of the goals for column must equal the sum of the goals against column, hence C had 3 goals against. A played three games and since at least one goal was scored in every game but one, A's scores were 1-0, 1-0, and 0-0. If C played only two games, the score of C's game that did not involve A was either 2-2 or 2-3, neither of which is possible. Hence C played three games.

Suppose D played three games. Then every team played three games. D's 3 games, similar to A's, must then have had scores of 0-0, 0-1, and 0-1. Therefore, A tied with D in the only game in which neither side scored. In B's games, since each involved at least one goal, the scores were 1-0, 1-0, and 0-1. Thus at most one goal was scored in every game involving either A, B, or D. Since every game involves A, B, or D, it follows that at most three goals were scored in C's games. This is a contradiction which shows D played at most two games.

Since D and B each played a game with A and one with C, we see that D played exactly two games. Thus D did not play B and B must have played only two games.

Therefore, we get the following:

Team	Games Played	Won	Lost	Drawn	Goals For	Goals Against
A	3	2	0	1	2	0
B	2				2	1
C	3			0	2	3
D	2				0	2

If D tied with A, then D's other game was a 0-2 loss, which is impossible. Therefore A tied with B and the results of A's games are

$$\begin{aligned} &A1 > D0 \\ &A1 > C0 \\ &A0 = B0 \end{aligned}$$

B's other game had a score of 2-1 which could only have been with C. It is now easy to find the complete results as shown in the following table: For each game, the score of the "row" team is given before the score of the "column" team.

A	B	C	D	
X	0-0	0-1	0-1	A
0-0	X	1-2	X	B
1-0	2-1	X	0-1	C
1-0	X	1-0	X	D

Challenge of the Week # 7 - March 23 to March 30: Martian soccer is a rather strange game. Three teams are involved in each game and it is played on a triangular field. Each team has a goal to defend and each team can score in either of the other two team's goals. The team scoring the highest number of goals in a game receives two points and the team scoring the second highest number receives one point. If there are two or three way ties, the points are shared between the teams that tie. On the martian web, the number of games a team has played, the points a team has earned, the number of goals a team has scored (goals for), and the number of goals opponents of a team have scored (opponent's goals) are listed. For example, if in the game involving the teams X, Y, and Z it happens that X and Y score 2 goals and Z scores none, the results would be X: 1.5 points, 2 goals for, 2 opponent's goals; Y: 1.5 points, 2 goals for, 2 opponent's goals; and Z: 0 points, 0 goals for, 4 opponent's goals. The following table was recently posted on http://N#%.mar giving some of the results of the final standings of last year's tournament involving A, B, C, D in which each triple of team played one game.

Team	*Games Played*	*Goals For*	*Opponent's Goals*	*Points Earned*
A		*2*	*2*	*4*
B	*3*	*1*	*4*	*2.5*
C		*0*		
D		*5*	*3*	*4.5*

Determine the results of each three way game and justify your answer.

Solved by Matt Wattleworth There are four possible triples and each team is involved in three of them. Hence, 12 points are earned. Since each goal is counted once in the Goals For column and twice in the Opponent's Goals column, the sum of the latter column is 16. This gives the following table:

Team	Games Played	Goals For	Opponent's Goals	Points Earned
A	3	2	2	4
B	3	1	4	2.5
C	3	0	7	1
D	3	5	3	4.5

Eight goals are scored by all teams and 8 were scored in matches involving D. Hence the score of the A,B,C match was A0-B0-C0 — a match in which each of the teams earned one point. By a similar argument, there are 4 goals scored in the B,C,D match, 3 goals scored in the A,C,D match, and 1 in the A,B,D match.

A earns one point in the A,B,C match; at most two in the A,C,D match and thus, at least one in the A,B,D match. The results of this match must have been A1-B0-D0. By looking at the Goals For column, we see that B scored a goal in the B,C,D match and C went scoreless. Since 4 goals were scored in this match, the results were D3-B1-C0. It now follows that the result of the remaining match was D2-A1-C0.

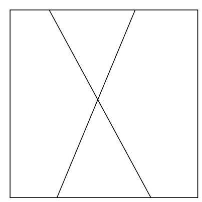
Challenge of the Week # 8 - March 30 to April 6: There are several parts to this special edition of the Challenge of the Week.

1. *A rooster's egg is placed on the top of any angled roof. On one side the roof angles down at* 30° *while on the other side the roof angles down at* 35°*. Which direction does the egg roll and why?*
2. *What is the maximum number of white rooks that can be placed on a chessboard so that no two of the rooks attack each other?*

3. *A chord is drawn in a circle so that its distance from the center is 1 cm. The length of this chord is 9.999 meters. How long is the chord whose distance from the center is 5 meters?*
4. *What is the fewest number of lines that can be drawn in order to divide a square into 2 pentagons and 2 triangles?*

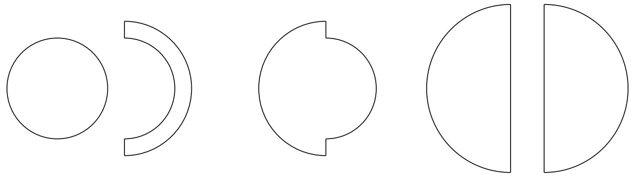
Solved by Peter Brusky, Will Cannon, Brandon Cox, Laura Duxbury, Julie Gommul, Roberta See, Erica Shifflet, and Adam Trant

1. Rooster's do not lay eggs.
2. White rooks do not attack each other.
3. Let r be the radius. Consider the right triangle composed of the segment from the center of the circle that is perpendicular to the chord, the segment from the point where this perpendicular intersects the chord to the circumference of the circle, and then the segment from the point on the circumeference to the center of the circle. We get
$$(\frac{9.999}{2})^2 + (0.01)^2 = r^2.$$
Using a calculator, it is easy to see that r is smaller than 5 meters. Hence there is no chord 5 meters from the center.
4. Solution without words.



Challenge of the Week # 9 - April 13 to April 20: Four people want to eat pancakes. But there are only three pancakes left - one 10 inches in diameter, one 8 inches in diameter and one 6 inches in diameter. Each person wants the same amount to eat. It is possible to divide up the pancakes into equal portions by dividing each cake in quarters and giving each person a quarter. This gives each person three pieces of cake. Show how to divide up the pancakes into equal portions so that three people get one piece of pancake and one person gets two pieces. The award will go to the simplest solution.

No simple solution was submitted for this week's challenge. Partial solutions were submitted by Peter Brusky, John Coon, and Kristin Laribee Cut the largest pancake it half. Place the smallest pancake so that the centers of the smallest and the middle pancake coincide. Now pick a diameter of the middle pancake. Cut, along the chosen diagonal, from a point on the circumference of the middle pancake to a point on the smallest pancake, then cut around the circumference of the smallest pancake to the other point where the chosed diagonal meets the smallest pancake. Finally cut out from the center of the pancakes along the chosen diagonal. The "half-moon"-shaped piece of the middle pancake plus the smallest pancake has the same area as the remaining pieces, i.e. they all have area $25\pi/2$. (See the diagram below.)



Challenge of the Week # 10 - April 20 to April 27: A treasure chest containing coins has been found. Some of the coins are gold, some are silver, some are copper, and some are bronze. It is discovered that every collection of 100 coins taken from the chest contains at least 10 coins of each type.

1. *Is it possible that there are exactly 150 coins in the chest?*
2. *What is the maximum number of coins there could be in the chest?*

Justify your answers.

There were no correct solutions submitted to this week's challenge.

1. It is not possible that there were 150 coins in the chest. Suppose there were 150 coins in the chest. It is not possible that there are more than $150/4 = 37.5$ coins of each type. Hence, there are at most 37 coins of one type and at least 113 coins of the other three types. Since $113 > 100$, there is a sample of 100 coins which has no coins of one type.

2. Let x, y, z, and w be the number of gold, silver, copper, and bronze coins, repectively. Let $n = x + y + x + w$ be the number of coins in the chest. In order that every sample contains at least 10 gold coins, $y + z + w \leq 90$. Similiarly,

$$\begin{aligned} x + z + w &\leq 90 \\ x + y + w &\leq 90 \\ x + y + z &\leq 90 \end{aligned}$$

   Adding these inequalities gives

$$3x + 3y + 3z + 3w \leq 360.$$

   Therefore, $n \leq 120$ and thus the maximum number of coins in the chest is at most 120. To see that the maximum is attained, consider the collection of 120 coins in which there are an equal number of each type of coin.