Challenges of the Week
Solutions
Fall Semester 1995-1996

Challenge of the Week # 1 - August 30 to September 8: You have a large container that contains a liter of milk and another large container which contains a liter of coffee. You take a one liter centiliter spoonful of milk out of the milk container and put it into the coffee container. You stir the coffee mixture until it is uniform. Then you take one centiliter of the mixture and put it into the milk. Is there more milk in the coffee container or more coffee in the milk container or are the two amounts the same?

Correct solutions were received from Wayne Clark, Amanda Ezell, Jason Nanoka, and Kamlesh Parwani. The amount of milk in the coffee container equals the amount of coffee in the milk container. Each of the correct solutions that were received showed that there were $\frac{100}{101}$ centiliters of coffee centiliters in the milk container and an equal amount of milk in the coffee container.

However, the problem could have been solved without using specific units. Suppose, originally, there is one unit of liquid in each container. Suppose after the liquids have been transferred there are m liters of milk in the coffee container. Since there is 1 liter of milk total, there must be $1 - m$ liters of milk in the milk container. This means that there are m liters of coffee in the milk container, as claimed.

Challenge of the Week # 2 - September 8 to September 15: You have four different letters, some of which weigh two ounces and some of which weigh three ounces. You go to the Post Office and wish to determine how much each letter weighs. At the Post Office, you find a scale. Clearly, you can determine the weight of each letter if you use four weighings and weigh each letter individually. What is the minimum number of weighings necessary to determine the weight of each letter? Justify your answer.

From the paper of Wayne Clark. It is possible to find the weight of each letter in less than four weighings. Mark one letter with an A, one with a B, one with a C, and the last one with an M (so as not to be predictable and boring). Your first step is to weigh A and B together. If they weigh 4 ounces, then each letter weighs 2 ounces. If they weigh 6 ounces, each letter weighs 3 ounces. In either case, you can determine the weight of C and M in two more weighings.

What if A and B weigh 5 ounces, you say? No problem. Next weigh A and C. If they weigh 4 ounces or 6 ounces, you can determine the weight of A, B and C. In one more weighing you can determine the weight of M.

OK, last possibility. You find that A and C weigh 5 ounces. Great. Only one weighing left. You might say you're out of luck. Then, I'd let you off the hook. "Weigh B, C, and M", I'd say. "Why?", you say. Well, it's simple. If they weigh six ounces, then B, C and M weigh 2 ounces each and A weighs 3 ounces. If they weigh 9 ounces, then B, C, and M weight 3 ounces each and A weighs 2 ounces. If they weigh 7 ounces, then B and C weigh 2 ounces, M weighs 3 ounces, and A weighs 3 ounces. (B and C weigh the same because A and C together weigh the same as A and B together. Cool, huh?) If the three letters weigh 8 ounces, then B and C weigh 3 ounces, M weighs 2 ounces, and A weighs 2 ounces. (That's all the possibilities since B, C, and M weigh between 6 and 9 ounces total.)

Challenge of the Week # 3 - September 15 to September 22: Suppose

$$x_1, x_2, \ldots, x_{1995}$$

are one thousand nine hundred and ninety five real numbers such that each is equal to the sum of the squares of the other 1994 numbers. Find all possible solutions for $x_1, x_2, \ldots, x_{1995}$.

From the paper of Kamlesh Parwani. Now,

$$\begin{aligned} x_1 &= x_2^2 + x_3^2 + \cdots + x_{1995}^2 \\ x_2 &= x_1^2 + x_3^2 + \cdots + x_{1995}^2 \end{aligned}$$

Subtracting these two equations gives $x_1 - x_2 = x_2^2 - x_1^2$. Therefore,

$$x_1 - x_2 = (x_2 - x_1)(x_2 + x_1).$$

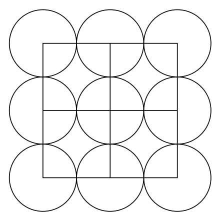
If $x_1 \neq x_2$, we can divide both sides of this last equation by $(x_1 - x_2)$ to get $-1 = x_2 + x_1$. This is impossible since x_1 and x_2 are both positive. Hence we must have $x_1 = x_2$.

A similar argument shows $x_i = x_j$ whenever $i, j = 1, 2, \ldots, 1995$. Let x be the common value. We then have $x = 1994x^2$. It follows easily that either $x = 0$ or $x = \frac{1}{1994}$.

Therefore, there are two solutions to the original equations — either $x_i = 0$ for all i or $x_i = \frac{1}{1994}$ for all i.

Challenge of the Week # 4 - September 22 to September 29: Pennies are arranged on a very, very large table (actually, the infinite plane). In the first pattern, each penny touches four other pennies and the straight lines joining the centers of pennies in contact with each other dissect the plane into equal squares. In the second pattern, each penny touches six others and the straight lines joining centers of pennies in contact with each other dissect the plane into equal, equilateral triangles.
Compute the percentage of the plane covered by the pennies for each pattern.

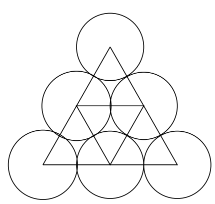
From the papers of Tina Mininni and Joe Nolan. Correct solutions were also received from Kamlesh Parwani and David Ruffatto. In part a), each penny touches four other pennies. Thus, by joining the centers of tangent pennies we see that the plane is divided into squares as shown below



The percentage of the area covered by the pennies in each square will equal the percentage covered over the entire table. Let r be the radius of a penny. There are four quarters of a penny in each square and the edge of each square is $2r$. Thus the percentage covered in this case is

$$\frac{\pi r^2}{4r^2} \times 100 = 25\pi \approx 78.54\%$$

In case b), we can join the centers of tangent pennies and get the following arrangement.



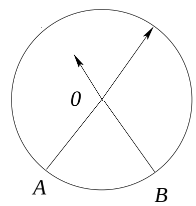
There are three sixths of a penny in each equilateral triangle for a total area of $\pi r^2/2$. Each edge of the equilateral triangle has length $2r$. Thus the area of the triangle if $\sqrt{3}r^2$. The percentage covered in this case is

$$\frac{\pi r^2/2}{r^2\sqrt{3}} \times 100 = \frac{50\pi}{\sqrt{3}} \approx 86.60\%$$

Challenge of the Week # 5 - September 29 to October 5: There are three hands on a large wall clock – an hour hand, a minute hand, and a second hand. A **good time** *is a time when the three hands are all in one-half of the clock face. For example, 1:15:30 is a good time. A* **bad time** *is a time when the three hands are not all in one-half of the clock face. For example, 3:50:30 is a bad time. In a 12 hour period are there more good times than bad times or more bad times than good times or are they equal? Justify your answer.*

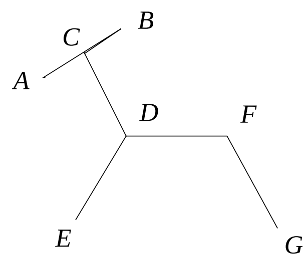
From the paper of Jason Nanoka. There are more good times than bad times. Consider, at first, only the second and the minute hand. Suppose these hands are at a certain position. These two hands will return to the same position in exactly one hour. Thus, the hour and minute hands are at this certain position exactly 12 times a day. Further, the correspoding positions of the hour hand divide the clock face into 12 equal parts.

Consider the 12 times where the angle between the second and the minute hand is θ. These two hands determine a sector AOB, as shown in the following figure, whose central angle is also θ.



If there are k bad times with this position for the second and minute hand, these times correspond to lines from the center 0 to points on arc AB. The central angles between these lines are $\pi/6$, since the twelve positions of the hour hand divide the clock face into 12 equal parts. Since the value of θ is at most π, there are at most 5 bad times with a specific position for the minute and second hand. It follows that there are more bad times in a 12 hour time period. Note: A rigorous proof of this problem would require calculus.

Challenge of the Week # 6 - October 6 to October 13: Five toothpicks of equal length are placed in the shape of a 'horse', as shown below. The angle determined by two toothpicks which touch is either $120°$ *or* $90°$*. Prove or disprove the following: It is possible to move one toothpick to another position so that the resulting figure is either congruent to the original 'horse' or is a mirror image of the 'horse'.*



From the papers of Samar B. Mitra and Kamlesh Parwani. Slide the toothpick with endpoints F and G in a direction perpendicular to this toothpick until the end that was at F coincides with E.

Challenge of the Week # 7 - October 13 to October 20: A square is drawn in the plane. Also in the plane is a point, which I can see, but you cannot see. This point is either inside the square or outside the square. In order to determine which is the case you can draw a sequence of lines on the plane. After you draw each line, I will tell you whether the point is on the line or not and, if not, will tell which side of the line the point is on. What is the minimum number of lines that you must draw in order to be certain that the point is inside the square or be certain that the point is outside the square? Justify your answer.

From the papers of Kamlesh Parwani and Joe Nolan. Three lines are necessary and sufficient. The determination cannot be done in two lines since two lines will not even allow one to isolate a finite area, let alone a square. To see that three lines are sufficient, let the vertices of the square be A, B, C, and D. The first line to be drawn is to pass through the diagonal AC. Without loss of generality, we may assume the point is either on this line or on the side of the line that contains B. Let the second line be drawn through A and B. If the point is on the side of the line which does not contain C or if the point is on this line, then the point must be outside the square. Hence we may assume the point is on the side of the line containing C. Now let the third line be drawn through B and C. If the point is on the side of this line that does not contain A or if the point is on this line, then the point is outside the square. If the point is on the side of this line that contains A, the point is inside the square.

Challenge of the Week # 8 - October 20 to October 27: If n *is a positive integer, let* $n^\star$ *denote the result of adding to* n *the sum of its digits. For instance,*

$$5^\star = 10, \qquad 86^\star = 100, \qquad 977^\star = 1000, \qquad 9968^\star = 10000.$$

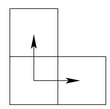
Find all numbers n *with* $n^\star = 100,000$*. Justify your answer.*

From the papers of Justin Large and Kamlesh Parwani. Also solved by Joe Nolan Suppose $n^\star = 100000$. Clearly, $n < 99999$ and thus the sum of the digits is less than 45. Therefore, n is between 99955 and 99999. Let $n = 99900 + 10x + y$, where $0 \leq x, y \leq 9$. We also have that

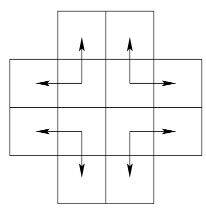
$$100000 = n^\star = 99900 + 27 + 11x + 2y.$$

That is, $73 = 11x + 2y$. Since $0 \leq y$, we have $11x \leq 73$ and $x \leq 6$. Since $y \leq 9$, we have $55 \leq 11x$ and $5 \leq x$. Finally $11x$ equals $73 - 2y$ which implies that both $11x$ and x are odd. Therefore, $x = 5$. It follows that $y = 9$ and $n = 99959$.

Challenge of the Week # 9 - October 27 to November 3: You are given a large number of L-shaped tiles, each consists of three unit squares arranged and marked as follows:



With four of these tiles you can cover a 4×4 checkerboard from which all four corners have been removed (as shown below).

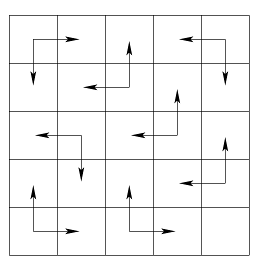


1. *Can a 5×5 checkerboard from which one corner square has been removed be covered by using L-shaped tiles?*
2. *Can a 6×6 checkerboard from which one corner square has been removed be covered?*
3. *Can an 8×8 checkerboard from which one corner square has been removed be covered?*
4. *If n is a positive integer, can a $2^n \times 2^n$ checkerboard from which one corner square has been removed be covered?*

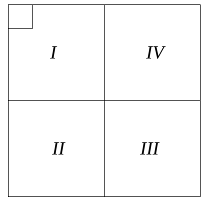
Justify your answers in each case.

Solutions from the papers of Kamlesh Parwani and Joe Nolan.

1. A 5×5 checkerboard with one corner removed can be covered as follows:



2. A 6×6 checkerboard with one corner removed has 35 squares. It is not possible to cover 35 squares with tiles each consisting of three squares.
3. Follows from part 4 of the problem
4. One L-shaped tile covers a 2×2 square with one corner removed. Suppose that you know how to cover a $2^k \times 2^k$ checkerboard with one square removed. Now take a $2^{k+1} \times 2^{k+1}$ checkerboard with one square removed. This larger checkerboard can be thought of as one $2^k \times 2^k$ checkerboard with a corner square removed and three $2^k \times 2^k$ checkerboards with no square removed. This larger board is represented in the following diagram which shows these four squares as well as the small square that is removed. The four squares are labelled for future reference.



To tile the larger checkerboard, first place a single tile so that it covers one square of each of checkerboards II, III, and IV. Now all four of the smaller checkerboards can be tiled by using the method required to tile a $2^k \times 2^k$ checkerboard with a corner square removed.

Challenge of the Week # 10 - November 3 to November 10: Two rooks, one white and one black, are placed on diagonally opposite corners of an 8×8 *chessboard. Both rooks move in the standard way, except neither can move on or across a row or column that is attacked by the other rook. The white rook goes first and play alternates. On each turn a player must move his/her rook - passing is not allowed. The first person who cannot make a legal move on his/her turn loses the game. With the best strategy, who wins the game, the player who plays first or the player who plays second? Justify your answer.*

From the papers of Jason Large, Lutful Khandker, and Kamlesh Parwani. For purposes of the solution, suppose the white rook is in the upper left corner and the black rook is in the lower right corner. Initially, the two rooks are on the same diagonal. On the first move, the white rook is moved off of the diagonal. The black rook may be moved to the left or upwards so that it is now on a common diagonal with the white rook. The winning strategy is a continuation of this where the black rook is always moved to the left or upwards so that it ends on a square that is diagonal from the square on which the white rook is located. Since the black rook is always moved upwards or to the left, the game must end. Since black always has a response (move to the diagonal square), the game must end with white having no move. In other words, with best stategy the second person will always win.