Solution - Challenge of the Week

Challenge of the Week # 7 - October 16 to October 23, 2009

Some number of different golden coins are put on the left pan of a balance, and some number of coins (which are all identical) are put on the right pan. Each coin on the right side has a larger diameter than any coin on the left pan, but the thickness of all the coins is the same. The balance is in equilibrium. Then each coin is replaced with a solid ball of the same diameter as the respective coin. All coins and all spheres are made of the same material. Will the balance be in equilibrium after that replacement? If not, which side will be heavier? Justify your answer.

Renee Fietsam provided a good solution to this week's challenge. Partial solutions were submitted by Katie Claypool, Rob Curtis, Chris Haines, Joseph Leipert, Nikki Miller, Brandon Morris, John Piccione, and Larry Trimble. Other submissions came from Chris DeSanto, Laura Kirchner, Ryan Steber, and Amanda Yingst

Solution: Let d be the density of the gold, t the thickness of the coins, n the number of coins on the left pan, r the radius of each coin on the left pan, and $s_1, s_2, \ldots, s_m$ be the radii of the coins on the right pan. Since the pans balance initially

$$dn\pi r^2 t = d\pi s_1^2 t + d\pi s_2^2 t + \cdots + d\pi s_m^2 t$$

or

$$nr^2 = s_1^2 + s_2^2 + \cdots + s_m^2.$$

After replacement, the weight of the spheres on the left pan is $dn\frac{4}{3}\pi r^3$ and the weight of spheres on the right pan is

$$d\frac{4}{3}\pi s_1^3 + d\frac{4}{3}\pi s_2^3 + \cdots d\frac{4}{3}\pi s_m^3.$$

As each coin on the right side has a larger diameter than each coin on the left side, $r < s_j$, $j = 1, 2, \ldots, m$. Hence,

$$\begin{aligned}
dn\frac{4}{3}\pi r^3 &= \frac{4d\pi}{3}(nr^2)r \\
&= \frac{4d\pi}{3}(s_1^2 + s_2^2 + \cdots + s_m^2)r \\
&= \frac{4d\pi}{3}(s_1^2 r + s_2^2 r + \cdots + s_m^2 r) \\
&< \frac{4d\pi}{3}(s_1^3 + s_2^3 + \cdots + s_m^3)
\end{aligned}$$

Thus, after replacement, the right hand side is heavier.