

## Solution - Challenge of the Week

### *Challenge of the Week # 6 - October 2 to October 16, 2009*

Let  $n$  be an integer.

1. Show that if  $n$  is not a multiple of three, then  $n^6 - 1$  is an integral multiple of 9.
2. Show that if  $n$  is not a multiple of two, then  $n^8 - 1$  is an integral multiple of 16.

**No correct solution was submitted for this week's challenge, although Joseph Leipert did submit a paper**

**Solution:**

1. If  $n$  is an integer that is not a multiple of three, then either  $n = 3k + 1$  or  $n = 3k - 1$ , for some integer  $k$ . Hence,  $n^2$  is either

$$(3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$$

or

$$(3k - 1)^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1.$$

In either case,  $n^2 = 3t + 1$ , for some integer  $t$ . Since

$$n^6 - 1 = (n^2 - 1)(n^4 + n^2 + 1),$$

when  $n^2$  is replaced by  $3t + 1$  we get

$$n^6 - 1 = ((3t + 1) - 1)((3t + 1)^2 + (3t + 1) + 1) = (3t)(9t^2 + 9t + 3) = 9(3t^3 + 3t^2 + t).$$

In particular,  $n^6 - 1$  is a multiple of 9.

2. If  $n$  is an integer that is not a multiple of two,  $n = 2k + 1$  for some integer  $k$ . Since

$$n^8 - 1 = (n - 1)(n + 1)(n^2 + 1)(n^4 + 1),$$

when  $n$  is replaced by  $2k + 1$  the result is

$$n^8 - 1 = ((2k + 1) - 1)((2k + 1) + 1)((2k + 1)^2 + 1)((2k + 1)^4 + 1).$$

Since each of these four factors is even, the product is divisible by 16, as needed.