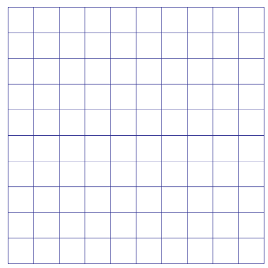
Solution - Challenge of the Week

Challenge of the Week # 4 - September 18 to September 25, 2009

A 10×10 grid contains squares of size 1×1, squares of size 2×2, ..., squares of size 9×9, and a single square of size 10×10.



What is the total number of squares of all sizes in a 10×10 grid? Justify your answer.

Complete solutions were provided by Hilary Cloe, Kristen Drozky, Jon Harter, Joseph Leipert, Alex Meadows, Katherine Mullins, Mollie Neff, Allen Obendorf, Jaimee Phegley, Michelle Raddatz, Josh Rappuhn, Fremont Schneider, Allysha Schuster, Ashlee Sharp, Dominique Sims, Dustin Trosper, and Cydnee Tucker. Other nice idea came from Mark Blount and William Petersen.

Solution: The number of squares of all sizes in an $n \times n$ grid is

$$1^2 + 2^2 + \cdots + n^2.$$

To see this consider an $n \times n$ grid and imagine the $(n-1) \times (n-1)$ grid in the upper left hand corner of the larger grid. Each 1×1 block in the $n \times n$ grid is the upper left hand corner 1×1 square in exactly one square (of some size) contained in the larger grid, but which is not contained in the $(n-1) \times (n-1)$ grid. Specifically, let A be a 1×1 square in the large grid. If you move diagonally down and right, you will eventually get to a 1×1 square, say B, on the boundary of the large grid, Now, A and B will be the upper left and lower right corners, respectively of a $k \times k$ square contained in the $n \times n$ grid, which is not in the $(n-1) \times (n-1)$ grid.

Thus, each time you increase the dimension of the grid by one, you add as many squares, of any size, as there are 1×1 squares in the larger grid. It follows that

$$1^2 + 2^2 + \cdots + n^2.$$

counts the number of squares of any size in an $n \times n$ grid

Finding the formula for the sum of squares can be a bid messy. Here is a method used by Leonardo of Pisa (Fibonacci) who lived flourished in the early 13th century

It is a simple matter to expand both sides to show, for any k,

$$k(k+1)(2k+1) = 6k^2 + (k-1)(k)(2k-1)$$

Hence,

$$\begin{array}{rcl} 1\cdot 2\cdot 3 & = & 6\cdot 1^2 \\ 2\cdot 3\cdots 4 & = & 6\cdot 2^2+1\cdot 2\cdot 3 \\ & \vdots & \\ (n-1)(n)(2n-1) & = & 6(n-1)^2+(n-2)(n-1)(2n-3) \\ n(n+1)(2n+1) & = & 6n^2+(n-1)(n)(2n-1). \end{array}$$

Adding all of these equations and cancelling terms which appear on both sides of the sum gives,

$$n(n+1)(2n+1) = 6(1^2+2^2+\cdots+n^2).$$

Therefore, the total number of squares of all sizes in a 10×10 grid is $\frac{(10)(11)(21)}{6} = 385$