Challenges of the Week
Solutions
Fall Semester 2001-2002

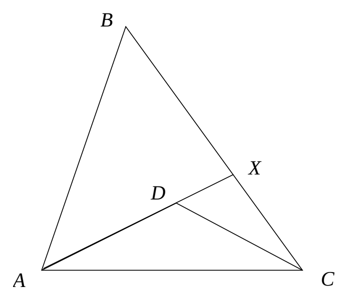
Challenge of the Week # 1 - August 24 to August 31: For X and Y points, let XY denote the distance between X and Y.

Suppose A, B, and C are vertices of a triangle. Let D be a point on the interior of △ABC. Show

$$AD + DC \leq AB + BC.$$

Under what conditions does equality hold?

No complete solution was submitted for this week's challenge. Let A, B, C, and D be as stated. Extend the line joining A to D until it meets the line joining B and C in a point X.



Since the shortest distance between two points is a straight line,

$$AB + BX \geq AX.$$

and

$$DX + XC \geq DC.$$

Further, since D is on the interior of the triangle, D is not on the line joining B and C. Hence, $DX + XC > DC$.

Therefore,

$$\begin{aligned} AB + BC &= AB + BX + XC \\ &\geq AX + XC \\ &= AD + DX + XC \\ &> AD + DC. \end{aligned}$$

Therefore, $AB+BC > AD+DC$, which means equality never holds. Note that if D is allowed to be on the boundary of $\triangle ABC$, then $AB + BC \geq AD + DC$, with equality only if $D = B$.

Challenge of the Week # 2 - September 7 to September 14: There are two tables in front you. One table contains thousands of quarters, exactly 31 of which are heads. The other table is empty. You are blind-folded and are wearing gloves, so you can't tell the difference between heads and tails. You are able to pick up coins, one at a time.

There are two types of moves you can make:

- *You can move a coin from one table to the other.*
- *You can turn a coin over.*

You can make as many moves of these types as you like. Can you eventually move the coins in such a way that there are the same number of heads on both tables?

Justify your answer.

Correct solutions for this week's challenge were received from Warren Buck and Nickolas Sloat Take 31 coins from the table with thousands of coins, turn them all over and then move them to the other table. Suppose y of the 31 coins that were transferred were heads. After the transfer, there would be $31 - y$ heads on the coins with thousands of coins. The other table contains 31 coins, y of them are tails and $31 - y$ of them are heads.

Challenge of the Week # 3 - September 14 to September 21, 2001: During the day, each of five different students learns a different bit of gossip about their teachers. In the evening, students begin calling each other, sharing the gossip. No two calls take place at the same time and no gossip is shared except by phone call. It is not hard to see that everyone can know all the gossip at the end of seven appropriately placed calls (See the table below which gives an example using seven calls. In the table, the bits of gossip known after each call are given. The students are denoted by A, B, C, D, E and the bit of gossip each knows initially is denoted by a, b, c, d, e, respectively.) Is it possible to arrange the calls so that all students know all pieces of gossip in fewer than seven calls?

	Gossip known				
	A	*B*	*C*	*D*	*E*
Initially	a	b	c	d	e
A calls B	a, b	a, b	c	d	e
B calls C	a, b	a, b, c	a, b, c	d	e
C calls D	a, b	a, b, c	a, b, c, d	a, b, c, d	e
D calls E	a, b	a, b, c	a, b, c, d	a, b, c, d, e	a, b, c, d, e
E calls A	a, b, c, d, e	a, b, c	a, b, c, d	a, b, c, d, e	a, b, c, d, e
A calls B	a, b, c, d, e	a, b, c, d, e	a, b, c, d	a, b, c, d, e	a, b, c, d, e
B calls C	a, b, c, d, e	a, b, c, d, e	a, b, c, d, e	a, b, c, d, e	a, b, c, d, e

Justify your answer.

Solved by Lydia Bruns, Colleen Camp, Kristy Mackovitch, Amy Miller, Symon Shmuckler, Missy Sikich, Gretchen Warner It is possible to pass all the information in only 6 calls, as is shown in the following table:

	Gossip known				
	A	B	C	D	E
Initially	a	b	c	d	e
A calls B	a,b	a,b	c	d	e
B calls C	a,b	a,b,c	a,b,c	d	e
D calls E	a,b	a,b,c	a,b,c	d,e	d,e
B calls E	a,b	a,b,c,d,e	a,b,c	d,e	a,b,c,d,e
C calls D	a,b	a,b,c,d,e	a,b,c,d,e	a,b,c,d,e	a,b,c,d,e
A calls C	a,b,c,d,e	a,b,c,d,e	a,b,c,d,e	a,b,c,d,e	a,b,c,d,e

It is possible to show that 5 calls are not sufficient. Note that ther are 5 "bits of knowledge" at the beginnning of the calls – A's knowledge of a, B's knowledge of b, ..., E's knowledge of e. At the end of the evening, there are 25 bits of knowledge – A, B, C, D, and E's knowledge of a, b, c, d, and e.

Suppose before the call of X to Y that X knows k rumors, Y knows m rumors, and they both know the same d rumors. Before the call, the two people together know $k + m - d$ bits of knowledge and after the call then know $2(k + m - d)$ bits of knowledge. Hence at most five bits of knowledge can be gained from each call.

On the first call, at most two bits of knowledge can be gained. On the second call, one person knows at most two rumors while the other knows at most one. Therefore at most three bits of knowledge can be gained from this call. After two calls, either four people each know two rumors each or, if one person was involved in both calls, one person, say W, knows three rumors while two others know their own rumor as well as w. Thus, at most four bits of knowledge can be gained from the third call.

Since at most five bits can be gained from each additional call, at most $5+2+3+4+5+5 = 24$ bits of knowledge are "known" after five calls.

Challenge of the Week # 4 - September 21 to September 28, 2001: Is it possible that the four sides and one diagonal of a quadrilateral have lengths, in some order, of 1, 2, 2.8, 5 and 7.5? If so, provide a proof. If not, construct an example.

Solved by Warren Buck, Jordan Hartigan, Kristy Mackovitch, and Symon Shmukler Recall that three segments can be used as the sides of a triangle if and only if the sum of the lengths of any two of them is larger than the length of the third.

In particular, there is a triangle with sides of lengths 1, 2, and 2.8 as well as a triangle with sides of lengths 2.8, 5, and 7.5. These two triangles can be placed together to form a quadrilateral.

This is the only way the lengths can be combined. For if the five lengths can be used as the lengths of the four sides and one diagonal of a quadrilateral, then 7.5 must be the length of one of the sides of a triangle formed from three of the five segments. This means that 2.8 and 5 are the lengths of the other two sides of this triangle. Since one of these lengths is also the length of a diagonal, the other triangle formed by the five segments has sides of length 1 and 2. This means the length of the diagonal is 2.8 and the quadrilateral is as described above. (Note: There are actually two non-congruent quadrilaterals with the desired properties!)

Challenge of the Week # 5 - October 3 to October 12: A magic square is an square array of numbers such that the sum of the entries in each row, column and diagonal is the same. Using the numbers 6, 8, 9, 10, 12, 13, 14, 15, 16, and 17, once each, complete the following to obtain a magic square.

	3		
	7		
4	*18*	*11*	
5			

Solved by Jason Antesberger, Karissa Berg, Danielle Borsilli, Lydia Bruns, Colleen Camp, Jenny Cassidy, Mark Cumbee Brian Esker, Matt Gresk, Lauren Hackett, Jill Hackler, Jennifer Hober, Cody Hollinshead, Lisa Judd, Neil Kirkpatrick, Lynn Kooistra, Trisha Kupscuk, Kristin Lavering, Jennifer Lutz, Kristy Mackovitch, Jason Madlem, Mike Ochs, Mark Palahniuk, Melissa Pales, Shawn Quigley, Kurt Ramsey, Christina Scales, Renée Shalris, Kyla Sigrist, Katie Springer, Jamie Uphoff, Andrew Weber, Gretchen Warner, Wenona Woolfolk, Qu Yunyi, and Robert Zerbst The numbers to be used are 3 through 18. Thus the sum of all numbers in the square is 168. Since this is also the sum of the numbers in the four rows, the sum of the numbers in each row, column, and diagonal is 42. The numbers in the third row and the second column are now determined.

	3		
	7		
4	18	11	9
5	14		

The numbers still to be assigned are 6, 8, 10, 12, 13, 15, 16 and 17. Since the sum of the two numbers to be placed in the first column is $42 - 9 = 33$, these numbers are 16 and 17. If 17 is placed in the upper left cell, then the lower right entry is determined as $42 - 35 = 7$, which is impossible. Thus 16 goes in the upper left cell. The entries in the first column and last row can now be determined.

16	3		
17	7		
4	18	11	9
5	14	15	8

Only one odd number, 13, remains. Since the sum of each row and column is even, this number must go in the first row and the last column. The entries in the square can now be easily determined.

16	3	10	13
17	7	6	12
4	18	11	9
5	14	15	8

Challenge of the Week # 6 - October 12 to October 19: In the context of this problem, ages are always rounded down. So, for example, if a person is 17 years and 364 days old, he or she is considered to be 17 (and therefore may not vote). Suppose Mrs. S was born on June 28th and had a daughter when she was 20 years old. The daughter was born on November 29th. When, if ever, was Mrs. S twice as old as her daughter?

Correct solutions were submitted by Lydia Bruns and Kevin Christian Let year X denote the year in which Mrs. S was 20 years old from June 28th until the end of the year. Let k be a positive integer. In year $X+k$, Mr. S was $20+k$ years old from June 28th until the end of the year and $20+k-1$ years old from January 1st until June 27th. In year $X+k$, her daughter had her k-th birthday on November 29th. Thus, in year $X+k$, her daughter was k years old from November 29th until the end of the year and $k-1$ years old from January 1st until November 28th.

Consider year $X+k$. There are three cases.

January 1 to June 27 Mrs. S will be twice as old as her daughter during this time when $20+k-1=2(k-1)$ or when $k=21$. At this time, Mrs. S will 40 and her daughter will be 20.

June 28 to November 28 Mrs. S will be twice as old as her daughter during this time when $20+k=2(k-1)$ or when $k=22$. At this time, Mrs. S will be 42 and her daughter will be 21.

November 29 to December 31 Mrs. S will be twice as old as her daughter during this time when $20+k=2k$ or when $k=20$. At this time, Mrs. S will be 40 and her daughter will be 20.

Thus, Mrs. S will be twice as old as her daughter from November 29, Year $X+20$, until June 27, Year $X+21$, and from June 28, Year $X+22$, until November 28 of the same year!

Challenge of the Week # 7 - October 19 to October 26: Another magic square: Complete the following array so that the sum of the numbers in each row, in each column, and in each diagonal is one. Give exact answers.

$\frac{1}{4}$		
		$\frac{3}{8}$

Solved by Lisa Anderson, Warren Buck, Jenny Cassidy, Kevin Christian, Cody Hollinshead, Jennifer Lutz, Andy McCormick, Colleen Murphy, Matt Ochs, Kurt Ramsey, Joy Rice, Kyla Sigrist, Robert Zerbst Let x denote the number in the center square. Consider the 12 entries in the two diagonals and the row and the column which contain the center square. The sum of these numbers is 4. On the other hand, each number of the array appears exactly once among these 12 entries, except for x which appears 4 times. Since the sum of the 9 numbers in the array is 3, we have

$$4 = 3 + 3x.$$

Thus $x=\frac{1}{3}$. The rest of the entries of the array are now determined and the array is

$\frac{1}{4}$	$\frac{13}{24}$	$\frac{5}{24}$
$\frac{7}{24}$	$\frac{1}{3}$	$\frac{3}{8}$
$\frac{11}{24}$	$\frac{1}{8}$	$\frac{5}{12}$

Challenge of the Week # 8 - October 26 to November 2: The following long-division problem, in which most of the digits are missing, comes out exactly. Fill in the missing digits. Justify your answer.

			2	__
__	__	__	__	__
		__	__	
		__	__	__
		3	__	__

Solved by Jason Antesberger, Donna Apke, Tom Becker, Jenny Cassidy, Kevin Christian, Matt Gresk, Cody Hollinshead, Mark Lumbee, Jennifer Lutz, Kristy Mackovitch, Andy McCormick, Todd Miller, Anna Morley, Matt Ochs, Shawn Quigley, Kurt Ramsey, Mike Schroeder, Eric Schrumpf, Symon Shmuckler, Nick Selner, Kyla Sigrist, Katie Springer, Miranda Stone, Gretchen Warner, Robert Zerbst Let x stand for the divisor, y stand for the two-digit number consisting of the leftmost digits of the dividend, and z stand for the last digit of the quotient. Since the leftmost digits in the next to last row is 3,

$$y - 2x \geq 30.$$

But $y \leq 99$. Since x must be an inte-ger, we get that $x \leq 34$. Also zx is a three-digit number whose leftmost digit is 3. Thus,

$$zx \geq 300$$

But $z \leq 9$ so

$$(*) \qquad 9x \geq zx \geq 300.$$

In particular, $9x \geq 300$. Since x is an integer, $x \geq 34$. Hence, $x = 34$. It follows from (*) that $z = 9$. Thus, the solution is

			2	9
3	4	9	8	6
		6	8	
		3	0	6
		3	0	6

Challenge of the Week # 9 - November 2 to November 9: An old classic: Queen Matilda, her son, and her daughter are captive in a very tall tower. Outside the window is a pulley on which there is a rope. Baskets are attached to either end of the rope. The rope is just long enough so either basket can be on the ground, while the other basket is outside the window.

The queen, who weighs 190 lb., her son, who weighs 170 lb., and her young daughter, who weighs 90 lb., find a 75 lb. stone in the room. The daughter can escape if she puts the stone in the basket. The stone and basket will fall to the ground bringing up the other basket. The daughter can then get in this basket and, since she weighs more than 75 lb., she will descend to the ground and can then get away.

Is it possible that the queen, daughter and son can all escape alive? Warning: If people try to descend in a basket in which the weight is more than 15 lb. more than the weight in the other basket, they will descend too rapidly and will not survive the fall. Justify your answer.

Correct solutions were received from Jason Antesberger, Brendan Aydt, Kristin Beer, Warren Buck, Colleen Camp, Jenny Cassidy, Kevin Christian, W. Clark Evers, Jill Hackler, Jordan Hartigan, Jennifer Hoben, Scott Lamb, Kristin Lovering, Kristy Mackovitch, Lauren Malinowski, Andy McCormick, Amy Miller, Anna Morley, Matt Ochs, Kurt Ramsey, Mike Schroeder, Nick Selner, Kyla Sigrist, Kristi Spray, Gretchen Warner, Andrew Weber, Wenona Woolfolk, Robert Zerbst All three can escape by using the procedure indicated in the following table.

Step	Weights in tower	Weights on ground
0.	75, 90, 170, 190	
1.	90, 170, 190	75
2.	75, 170, 190	90
3.	170, 190	75, 90
4.	75, 90, 190	170
5.	90, 190	75, 170
6.	75, 190	90, 170
7.	90, 170	75, 190
8.	75, 170	90, 190
9.	170	75, 90, 190
10.	75, 90	170, 190
11.	90	75, 170, 190
12.	75	90, 170, 190
13.		75, 90, 170, 190

Challenge of the Week # 10 - November 9 to November 16: You have four coins, denoted A, B, C, and D. Also, you have a pan balance on which you can compare the weights of coins. However, there is friction in the mechanism and as a result two coins whose weight is nearly the same will balance. Coins B and D balance. Coins C and D balance. However, B is definitely heavier than C and D is definitely lighter than A. Is it possible to arrange the coins in order of increasing weight? Justify your answer.

Solved by Tom Becker, Jenny Cassidy, Colleen Camp, Matt Gresk, Nancy Grove, Jill Hackler, Cody Hollinshead, Neil Kirkpatrick, Kristin Lovering, Kristy Mackovitch, Andy McCormick, Anna Morley, Mike Ochs, Joy Rice, Kyla Sigrist, Krist Spray, Gretchen Warner, Andrew Weber, and Robert Zerbst In the following, the letter denoting the coin is also used to denote the weight of the coin. It is given that $C < B$. If $D \leq C$, then D and B would not balance. Since they do balance, $C < D$. Because $C < B$ and C and D balance, a similar argument show $D < B$. Thus, even thought D seems to balance with either C or B, $C < D < B$. Finally, it is given that D is definitely lighter than A. Since D is also lighter than B, even though the balance does not show this, $B < A$. Therefore, $C < D < B < A$.