$\LaTeX$ samples from various area of mathematics

Calculus

Find the following limit: $\lim_{x\to -1}\frac{x^2 - 2x - 3}{x^2 - 1}$.

Solution: Now, $\frac{x^2-2x-3}{x^2-1} = \frac{(x-3)(x+1)}{(x-1)(x+1)}$. This means,

$$\frac{x^2-2x-3}{x^2-1} = \frac{x-3}{x-1}, \mbox{ provided } x \neq -1.$$

Therefore,

$$\lim_{x\to -1}\frac{x^2 - 2x - 3}{x^2 - 1} = \lim_{x\to -1}\frac{x-3}{x-1} = \frac{-4}{-2} = 2.$$

Source

```
Find the following limit:
$\lim_{x\to -1}\frac{x^2 - 2x - 3}{x^2 - 1}$.

\textbf{Solution:} Now,
$\frac{x^2-2x-3}{x^2-1} = \frac{(x-3)(x+1)}{(x-1)(x+1)}$.  This means,

\[\frac{x^2-2x-3}{x^2-1} = \frac{x-3}{x-1}, \mbox{ provided } x \neq -1.\]

Therefore,

\[\lim_{x\to -1}\frac{x^2 - 2x - 3}{x^2 - 1} =
  \lim_{x\to -1}\frac{x-3}{x-1} = \frac{-4}{-2} = 2.\]
```

Linear Algebra

Let $L\colon \mathbb{R}^4\to \mathbb{R}^4$ be the linear transformation defined by

$$L\left(\left[\begin{array}{c} a_1\\a_2\\a_3\\a_4 \end{array}\right]\right) = \left[\begin{array}{cccc} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{c} a_1\\a_2\\a_3\\a_4 \end{array}\right]$$

Find a basis for $\mbox{ker } L$.

Solution:

To find a basis for $\mbox{ker } L$, we want to find a basis for the nullspace of the matrix

$$\left[\begin{array}{cccc} 1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1 \end{array}\right].$$

Source

```
Let $L\colon \mathbb{R}^4\to \mathbb{R}^4$ be the linear transformation defined by
\[L\left(\left[
    \begin{array}{c}
        a_1\\a_2\\a_3\\a_4
    \end{array}
 \right]\right)
  =
 \left[
    \begin{array}{cccc}
        1 & 1 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1
    \end{array}
 \right]
 \left[
    \begin{array}{c}
        a_1\\a_2\\a_3\\a_4
    \end{array}
 \right]\]

Find a basis for $\mbox{ker } L$.

\textbf{Solution:}

To find a basis for $\mbox{ker } L$, we want to find a basis for
the nullspace of the matrix
\[\left[\begin{array}{cccc}
     1 & 1 & 0 & 0\\
     0 & 1 & 1 & 0\\
     0 & 0 & 1 & 1\\
     1 & 0 & 0 & 1
   \end{array}\right].\]
```

Abstract Algebra

Show that $G = \{2k | k \in \mathbb{Z}\}$ and $H = \{3k | k \in \mathbb{Z}\}$ are isomorphic (as additive groups).

Solution: Let $\alpha\colon G \to H$ be defined by $\alpha(2k) = 3k$, $k \in \mathbb{Z}$.

Since every element of G is even and every element of H is a multiple of 3, α is a map from G to H.

α is injective: If $2k_1, 2k_2$ are elements of G with $\alpha(2k_1) = \alpha(2k_2)$, then $3k_1 = 3k_2$. Dividing both sides by 3 and multiplying by 2 gives $2k_1 = 2k_2$, as needed.

α is surjective: Let $h \in H$. Then $h = 3t$ for some $t \in \mathbb{Z}$. Since $2t \in G$ and $\alpha(2t) = 3t = h$, α is surjective.

α is operation preserving: Let $2k_1$ and $2k_2$ be elements of G, then

$$\alpha(2k_1 + 2k_2) = \alpha(2(k_1 + k_2)) = 3(k_1 + k_2) = 3k_2 + 3k_2 = \alpha(2k_1) + \alpha(2k_2).$$

Source

```
Show that $G = \{ 2k | k\in \mathbb Z \}$ and
$H = \{ 3k | k \in \mathbb Z \}$
are isomorphic (as additive groups).

\textbf{Solution:}
Let  $\alpha\colon G\to H$ be defined by
$\alpha(2k) = 3k$, $k\in\mathbb Z$.

Since every element of $G$ is even and every element of $H$ is a
multiple of 3, $\alpha$ is a map from $G$ to $H$.

$\alpha$ is injective:  If $2k_1, 2k_2$ are elements of $G$ with
$\alpha(2k_1) = \alpha(2k_2)$, then $3k_1 = 3k_2$.  Dividing both sides
by 3 and multiplying by 2 gives $2k_1 = 2k_2$, as needed.

$\alpha$ is surjective:  Let $h \in H$.  Then $h = 3t$ for some $t\in
\mathbb Z$.  Since $2t \in G$ and $\alpha(2t) = 3t = h$, $\alpha$ is surjective.

$\alpha$ is operation preserving:  Let $2k_1$ and $2k_2$ be elements
of $G$, then
\[
 \alpha(2k_1 + 2k_2) =
 \alpha(2(k_1 + k_2)) =
 3(k_1 + k_2) = 3k_2 + 3k_2 =
 \alpha(2k_1) + \alpha(2k_2).
\]
```

Probability

Given the independent random variables $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ with probability densities

$$\begin{aligned}
f_1(x_1) &= \left\{\begin{array}{ll} e^{-x_1} & \mbox{for } x_1 > 0\\ 0 & \mbox{elsewhere} \end{array}\right.\\
f_2(x_2) &= \left\{\begin{array}{ll} 2e^{-2x_1} & \mbox{for } x_1 > 0\\ 0 & \mbox{elsewhere} \end{array}\right.\\
f_3(x_3) &= \left\{\begin{array}{ll} 3e^{-3x_1} & \mbox{for } x_1 > 0\\ 0 & \mbox{elsewhere} \end{array}\right.
\end{aligned}$$

find their joint probability density, and use it to evaluate the probability $P(\mathbf{x}_1 + \mathbf{x}_2 \le 1,\ \mathbf{x}_3 > 1)$.

Solution: The joint probability density function is $f(x_1, x_2, x_3) = 6e^{-x_1-2x_2-3x_3}$ for x_1, x_2, and x_3 all positive and is zero elsewhere. Thus,

$$\begin{aligned}
P(\mathbf{x}_1 + \mathbf{x}_2 \le 1, \mathbf{x}_3 > 1) &= \int_1^\infty \int_0^1 \int_0^{1-x_2} 6e^{-x_1-2x_2-3x_3} dx_1\ dx_2\ dx_3\\
&= (1 - 2e^{-1} + e^{-2})e^{-3}\\
&\approx 0.020
\end{aligned}$$

Source

```
Given the independent random variables $\textbf{x}_1$, $\textbf{x}_2$,
and $\textbf{x}_3$ with probability densities

\begin{eqnarray*}
f_1(x_1) &=&
  \left\{\begin{array}{ll}
     e^{-x_1}  & \mbox{for } x_1 > 0\\
     0 &\mbox{elsewhere}
  \end{array}\right.\\
f_2(x_2) &=&
  \left\{\begin{array}{ll}
     2e^{-2x_1}  & \mbox{for } x_1 > 0\\
     0 &\mbox{elsewhere}
  \end{array}\right.\\
f_3(x_3) &=&
  \left\{\begin{array}{ll}
     3e^{-3x_1}  & \mbox{for } x_1 > 0\\
     0 &\mbox{elsewhere}
  \end{array}\right.\\
\end{eqnarray*}

find their joint probability density, and use it to evaluate the
probability $P(\textbf{x}_1 + \textbf{x}_2 \le 1, \textbf{ x}_3 > 1)$.

\textbf{Solution:}  The joint probability density function is
$f(x_1, x_2, x_3) = 6e^{-x_1 - 2x_2 - 3x_3}$  for $x_1$, $x_2$, and
$x_3$ all positive and is zero elsewhere.  Thus,

\begin{eqnarray*}
P(\textbf{x}_1 + \textbf{x}_2 \le 1, \textbf{x}_3 > 1)
  &=& \int_1^\infty \int_0^1 \int_0^{1 - x_2} 6e^{-x_1 - 2x_2 - 3x_3} dx_1\ dx_2\ dx_3\\
  &=& (1 - 2e^{-1} + e^{-2})e^{-3}\\
  &\approx& 0.020
\end{eqnarray*}
```