Alternating Knots & Montesinos Knots Satisfy the (Classical) L-space Surgery Conjecture

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Foliations

A *foliation* is a decomposition of a manifold into *leaves* of lower dimension. Locally, we have charts $\mathbb{R}^m \times \mathbb{R}^n$, with transitions that preserve the horizontal levels $\mathbb{R}^m \times \{y\}$.

We consider foliations of smooth 3-manifolds with 2-dimensional $C^1$-embedded leaves (co-dimension 1).
Taut Foliations

Definition

A co-dimension 1 foliation of a 3-manifold is *taut* if there is a circle transversely intersecting every leaf.

*Remark:* A closed manifold admitting a taut foliation is universally covered by $\mathbb{R}^3$, hence is irreducible and has infinite fundamental group.

Definition

A 3-manifold is *foliar* if it admits a taut, co-orientable (co-dimension 1) foliation.
Heegaard-Floer Homology

An homology theory for rational homology 3-spheres.

- Introduced by P. Ozsváth & Z. Szabó.
- $\widehat{HF}(M)$ is a vector space over $\mathbb{F}_2$.
- $\text{Rank}(\widehat{HF}(M)) \geq |H_1(M, \mathbb{Z})|$.
- If equality holds, $M$ is an $L$-space.
- $L$-spaces include lens spaces.

Theorem (Eliashberg–Thurston, Ozsváth–Szabó, Kazez-Roberts)

$M$ admits a taut, co-orientable foliation $\Rightarrow$ $M$ is not an $L$-space

Does the converse hold for irreducible 3-manifolds?
(Ozsváth–Szabó, Boyer-Gordon-Watson, Juhasz?)
A (classical) knot is an \((n - 2)\)-sphere embedded in an \(n\)-sphere, in particular, for \(n = 3\).

\[
\text{Knot in } S^3
\]

(alternating)

Note that a \textit{regular neighborhood} (“fattening up”) of a knot is a solid torus.
Dehn Surgery

- Remove a solid torus (a “fattened up” knot) from $S^3$ and glue in a solid torus by a homeomorphism of $T^2$.
- The result depends only on the curve to which the meridian is glued.
- $l$ longitudes and $m$ meridians, $l, m$ relatively prime, give Dehn surgery coefficient $\frac{m}{l} \in \mathbb{Q} \cup \frac{1}{0}$.
- Coefficient $1/0$ is trivial surgery (yielding $S^3$ back).
Two Interesting Types of Knots

- In particular, we consider two classes of knots:
  - Alternating knots
  - Montesinos knots:

\[ M(1/3, 2/5, 3/5, -1) \]

- The pretzel knots are a subset of the Montesinos knots:

\[ (3,3,3) \text{-Pretzel Knot} \]
**Terminology**

**Definition**

A knot $k$ is *persistently foliar* if every manifold obtained by non-trivial Dehn surgery on $k$ is foliar.

**Definition**

A knot $k$ is an *L-space knot* if some non-trivial surgery on $k$ yields an L-space.

**Corollary**

*If a knot is persistently foliar, it is not an L-space knot.*
Conjectures [D-Roberts]

Restricting attention to surgery on knots $k \subset S^3$, we conjecture the following:

**L-space Knot Conjecture** If $k$ does not admit a non-trivial reducible or L-space surgery, then $k$ is persistently foliar.

More generally,

**L-space Surgery Conjecture** A manifold obtained by Dehn surgery on $k$ is foliar if and only if it is irreducible and not an L-space.
Results

Theorem (D-Roberts)

All alternating knots satisfy the L-space surgery conjecture. In particular, every non-torus alternating knot is persistently foliar.

Remark: For torus knots, the result follows from the classification of their foliar (Boyer, Eisenbud-Hirsch-Neumann, Jenkins-Neumann, Raimi) and L-space (Hedden) surgeries.

Theorem (D-Roberts)

All Montesinos knots satisfy the L-space surgery conjecture. In particular, every Montesinos knot that is not an L-space knot is persistently foliar.

Remark: The result for L-space knots follows from work of Baker, Lidman, Hedden, Moore, and Roberts.
Finite Depth Spines

- Build a spine (Casler) from a finite succession of transversely intersecting surfaces.
- Locally:

  Surface neighborhood

  Double point neighborhood

  Triple point neighborhood
Smoothing Instructions

- Successively introduce *smoothing instructions* at singular points to obtain a *branched surface* (continuous tangent plane field):

  - Surface neighborhood
  - Double point neighborhood
  - Triple point neighborhood

- Eventually obtain a transversely orientable laminar branched surface for which the complement of an $I$-bundle neighborhood is a taut sutured manifold.
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I-bundle Neighborhood

Surface neighborhood
Double point neighborhood
Triple point neighborhood
- **Arrow-diamond** notation at a double point with one distinguished sector:
There are 12 possible smoothings at a triple point:
Work in the Knot *Exterior*

- Work in the knot exterior: $S^3 \setminus K$
- Introduce a “tube” around $K$: $T = \partial N(K) \subset S^3 \setminus K$
- $T$ is part of the spine.
- Convention: Outward normal to $T$ points into knot complement, out of $N(K)$.
Meridional Cusps $\rightarrow$ Persistence

Goal:

- Build spine having meridional intersections with $T$.
- Smooth to branched surface $\Sigma$ with even ($>0$) number of meridional branch curves with outward sink direction on $T$.
- After any rational Dehn surgery, these yield an even number of longitudinal sutures, so a meridional disk fully decomposes $N(K')$ (as a taut sutured manifold).
Meridional Cusps → Persistence (continued)

- Thus, as long as the other components of $\mathcal{N}(\Sigma)^c$ are taut sutured manifolds, we obtain a taut co-orientable foliation in every manifold produced by (non-trivial) surgery.

- This is what we mean by *persistence*.

- Antecedent: “Swallow-follow” closed (branched) surface. (Menasco; Oertel)
Method 1: Decomposition by Spheres & Spanning Surfaces

- Decompose $K$ into tangles along transverse spheres.
- Decompose further along spanning surfaces for the tangles.
- Similar to Murasugi sum, but surfaces on each side need not match.

With suitable choices, we obtain persistence, and every component of $\mathcal{N}(\Sigma)^c$ is a taut sutured manifold.
Example: Branched Surfaces in the Complement of $T(1/3)$

- Branched surfaces ↔ paths in the Farey diagram.
- From outside the tangle, we see a twisted band.
Channel Branched Surface: Level Set Sequence
Channel Branched Surface

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View from above

View from below

Triple point

Meridional cusp

Interior view
Combining Rational Tangles: the Enveloping Surface

The $(3, 3, 3)$ pretzel knot, $K(1/3, 1/3, 1/3)$, is persistently foliar!
Application of Method 1

- Method 1 works well for Montesinos knots, since they decompose into rational tangles.
- Method 1 shows all Montesinos knots to be persistently foliar except for some “small” pretzel knots.
Method 2: Decomposition of a Spanning Surface

May be viewed as a generalization of Gabai’s theory:

Sutured manifold decomposition of a Seifert surface

Generalized surface decomposition of a spanning surface
Some Differences; Application

Generalized decomposition of a spanning surface provides much greater flexibility:

- Persistence.
- Initial spanning surface need not be orientable!
- Boundary of decomposing surface can cross over $T$ from one side of $S$ to the other an odd number of times!

Method 2 shows all non-torus alternating knots and all remaining pretzel knots that are not $L$-space knots to be persistently foliar.
Local Models and Notation Conventions: Type A

With positive twist:

(Source)

(Sink)
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Local Models and Notation Conventions: Type A
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Local Models and Notation Conventions: Type B

With positive twist:

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Local Models and Notation Conventions: Type C

With positive twist:
The \((-2, 5, 5)\) Pretzel Knot is Persistently Foliar
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Sample Disk Decompositions in the Alternating Setting
Questions?
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Thank you!