Alternating Knots & Montesinos Knots Satisfy the (Classical) L-space Surgery Conjecture

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Joint work with Rachel Roberts

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Foliations

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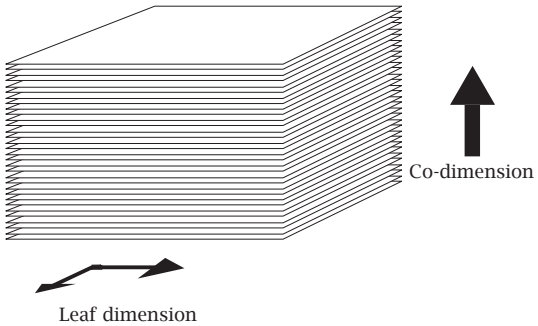
A *foliation* is a decomposition of a manifold into *leaves* of lower dimension. Locally, we have charts $\mathbb{R}^m \times \mathbb{R}^n$, with transitions that preserve the horizontal levels $\mathbb{R}^m \times \{y\}$.



We consider foliations of smooth 3-manifolds with 2-dimensional C^1-embedded leaves (co-dimension 1).

Taut Foliations

Definition

A co-dimension 1 foliation of a 3-manifold is *taut* if there is a circle transversely intersecting every leaf.

Remark: A closed manifold admitting a taut foliation is universally covered by $\mathbb{R}^3$, hence is irreducible and has infinite fundamental group.

Definition

A 3-manifold is *foliar* if it admits a taut, co-orientable (co-dimension 1) foliation.

Heegaard-Floer Homology

An homology theory for rational homology 3-spheres.

- Introduced by P. Ozsváth & Z. Szabó.
- $\widehat{HF}(M)$ is a vector space over $\mathbb{F}_2$.
- $\text{Rank}(\widehat{HF}(M)) \geq |H_1(M, \mathbb{Z})|$.
- If equality holds, M is an *L-space*.
- L-spaces include lens spaces.

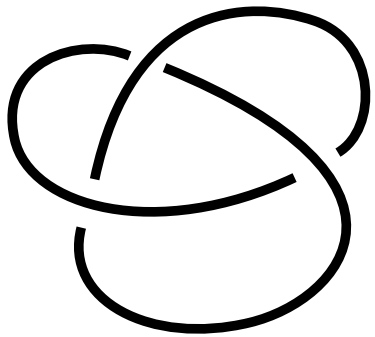
Theorem (Eliashberg–Thurston, Ozsváth–Szabó, Kazez-Roberts)

M admits a taut, co-orientable foliation $\Rightarrow$ M is not an L-space

Does the converse hold for irreducible 3-manifolds? (Ozsváth–Szabó, Boyer-Gordon-Watson, Juhasz?)

Knots

A (classical) knot is an $(n-2)$-sphere embedded in an n-sphere, in particular, for $n = 3$.



Knot in S^3
(alternating)

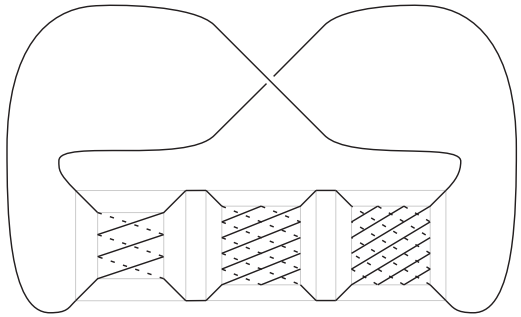
Note that a *regular neighborhood* ("fattening up") of a knot is a solid torus.

Dehn Surgery

- Remove a solid torus (a "fattened up" knot) from S^3 and glue in a solid torus by a homeomorphism of T^2.
- The result depends only on the curve to which the meridian is glued.
- l longitudes and m meridians, l, m relatively prime, give *Dehn surgery coefficient* $\frac{m}{l} \in \mathbb{Q} \cup \frac{1}{0}$.
- Coefficient $1/0$ is trivial surgery (yielding S^3 back).

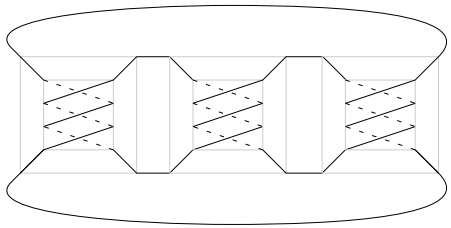
Two Interesting Types of Knots

- In particular, we consider two classes of knots:
  - Alternating knots
  - *Montesinos* knots:



M(1/3, 2/5, 3/5, -1)

  - The *pretzel* knots are a subset of the Montesinos knots:



(3,3,3)-Pretzel Knot

Terminology

Definition

A knot k is *persistently foliar* if every manifold obtained by non-trivial Dehn surgery on k is foliar.

Definition

A knot k is an *L-space knot* if some non-trivial surgery on k yields an L-space.

Corollary

If a knot is persistently foliar, it is not an L-space knot.

Conjectures [D-Roberts]

Restricting attention to surgery on knots $k \subset S^3$, we conjecture the following:

L-space Knot Conjecture *If k does not admit a non-trivial reducible or L-space surgery, then k is persistently foliar.*

More generally,

L-space Surgery Conjecture *A manifold obtained by Dehn surgery on k is foliar if and only if it is irreducible and not an L-space.*

Results

Theorem (D-Roberts)

All alternating knots satisfy the L-space surgery conjecture. In particular, every non-torus alternating knot is persistently foliar.

Remark: For torus knots, the result follows from the classification of their foliar (Boyer, Eisenbud-Hirsch-Neumann, Jenkins-Neumann, Raimi) and L-space (Hedden) surgeries.

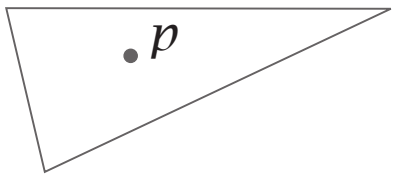
Theorem (D-Roberts)

All Montesinos knots satisfy the L-space surgery conjecture. In particular, every Montesinos knot that is not an L-space knot is persistently foliar.

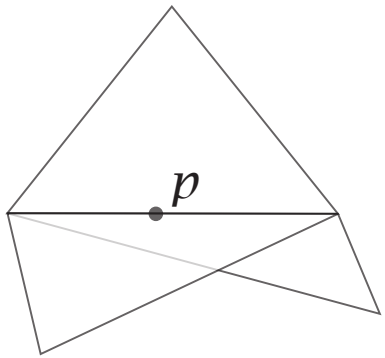
Remark: The result for L-space knots follows from work of Baker, Lidman, Hedden, Moore, and Roberts.

Finite Depth Spines

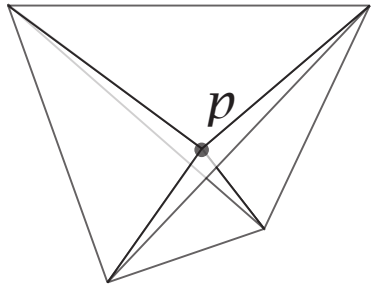
- Build a *spine* (Casler) from a finite succession of transversely intersecting surfaces.
- Locally:



Surface neighborhood



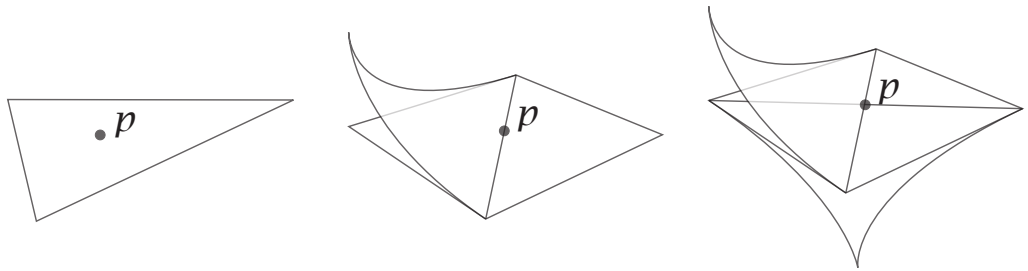
Double point neighborhood



Triple point neighborhood

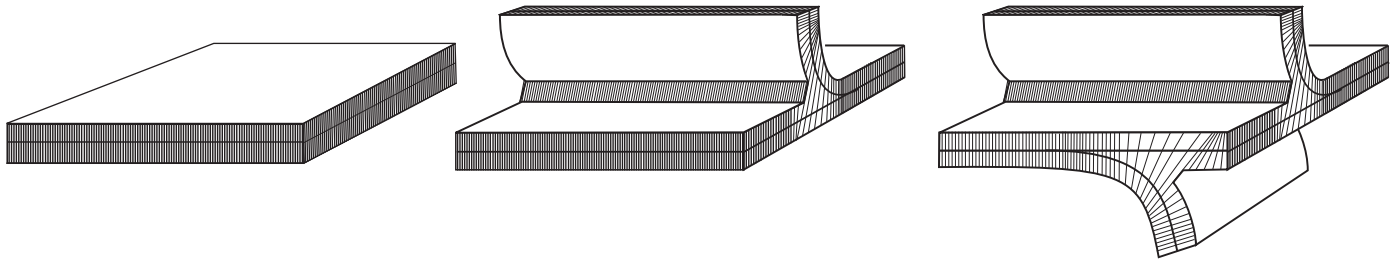
Smoothing Instructions

- Successively introduce *smoothing instructions* at singular points to obtain a *branched surface* (continuous tangent plane field):



- Eventually obtain a transversely orientable laminar branched surface for which the complement of an I-bundle neighborhood is a taut sutured manifold.

I-bundle Neighborhood



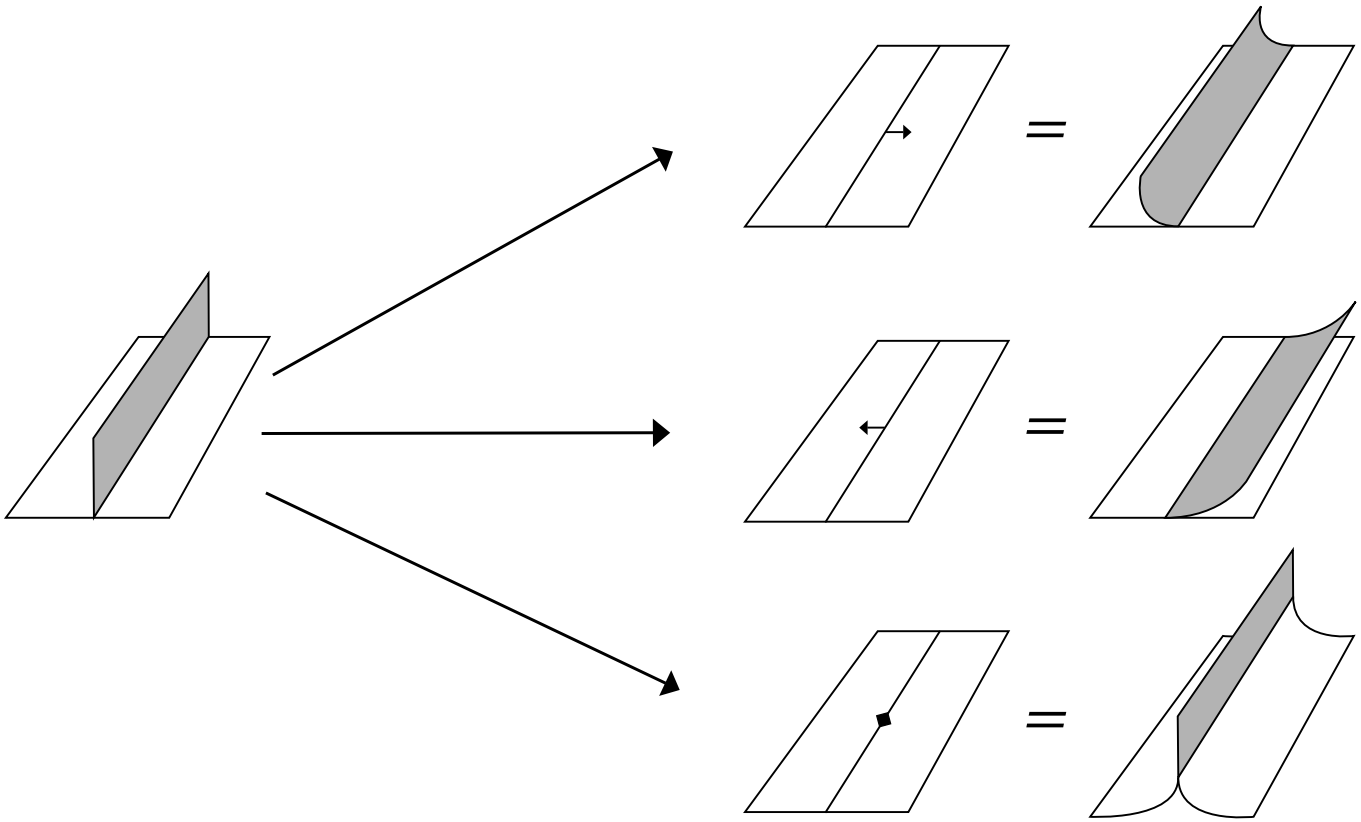
Surface neighborhood

Double point neighborhood

Triple point neighborhood

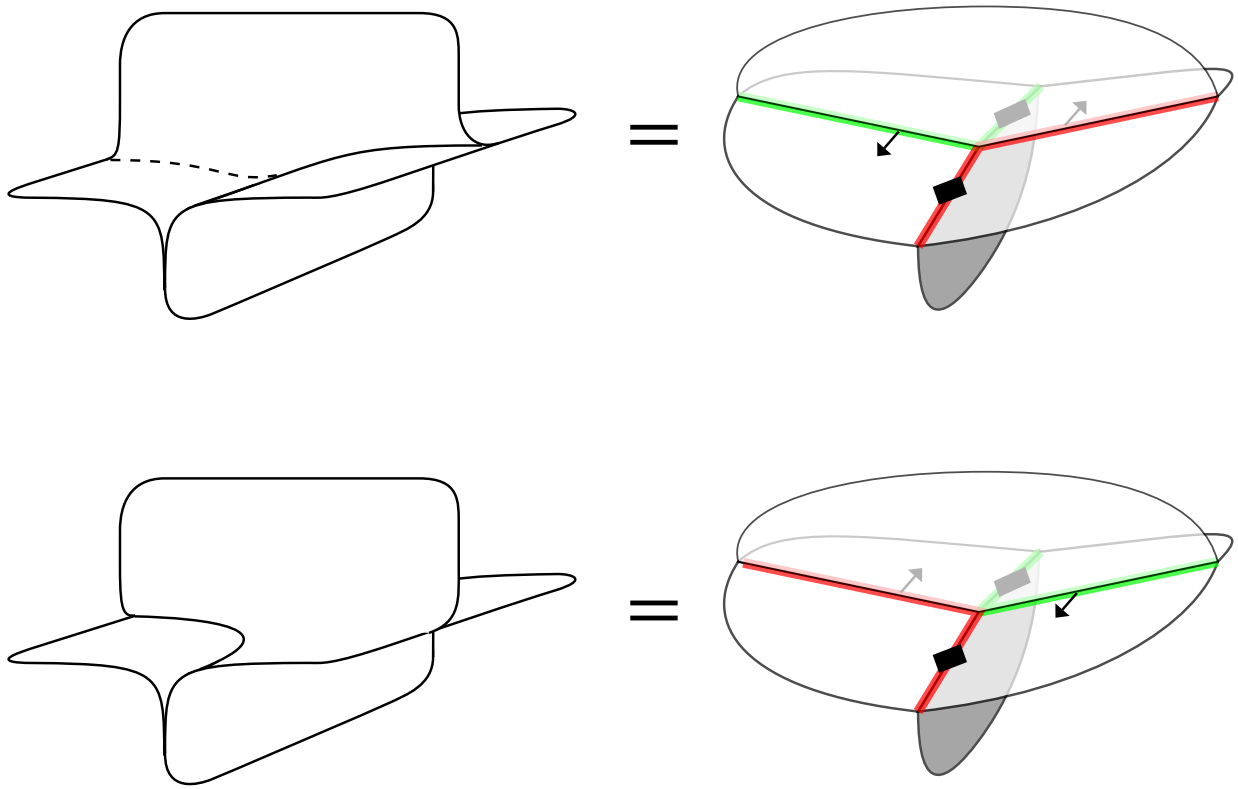
Notation

- *Arrow-diamond* notation at a double point with one distinguished sector:



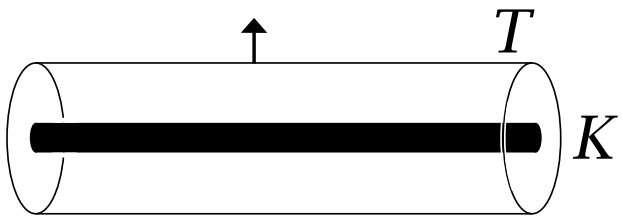
Smoothings at a Triple Point

- There are 12 possible smoothings at a triple point:



Work in the Knot *Exterior*

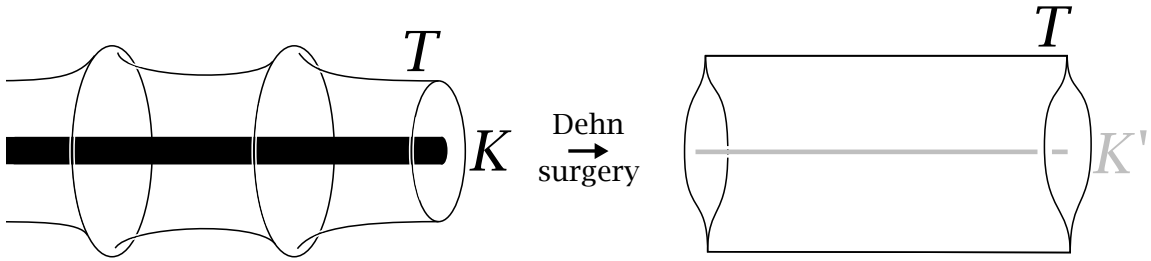
- Work in the knot *exterior:* $S^3 \setminus K$
- Introduce a "tube" around K: $T = \partial N(K) \subset S^3 \setminus K$
- T is part of the spine.
- Convention: Outward normal to T points into knot complement, out of $N(K)$.



Meridional Cusps $\rightarrow$ Persistence

Goal:

- Build spine having meridional intersections with T.
- Smooth to branched surface Σ with even (> 0) number of meridional branch curves with outward sink direction on T.
- After any rational Dehn surgery, these yield an even number of longitudinal sutures, so a meridional disk fully decomposes $N(K')$ (as a taut sutured manifold).

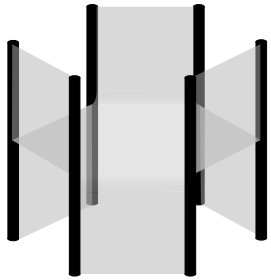


Meridional Cusps $\rightarrow$ Persistence (continued)

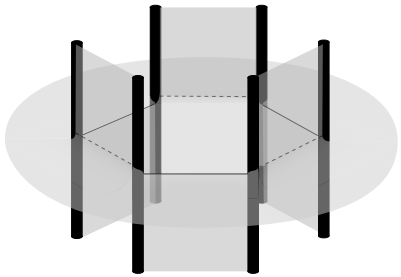
- Thus, as long as the other components of $\overline{N(\Sigma)^c}$ are taut sutured manifolds, we obtain a taut co-orientable foliation in every manifold produced by (non-trivial) surgery.
- This is what we mean by *persistence*.
- Antecedent: "Swallow-follow" closed (branched) surface. (Menasco; Oertel)

Method 1: Decomposition by Spheres & Spanning Surfaces

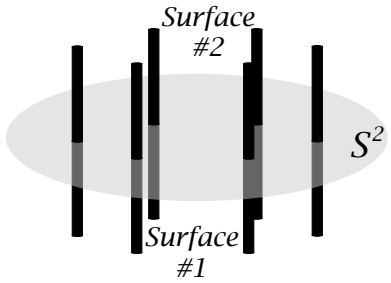
- Decompose K into tangles along transverse spheres.
- Decompose further along spanning surfaces for the tangles.
- Similar to Murasugi sum, but surfaces on each side need not match.



Murasugi sum



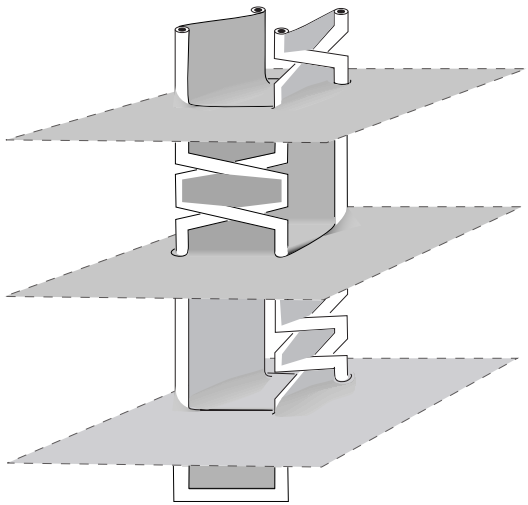
Branched Murasugi sum

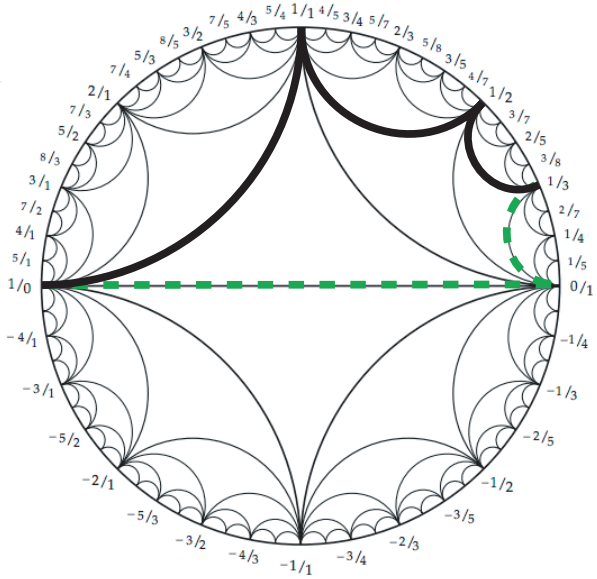


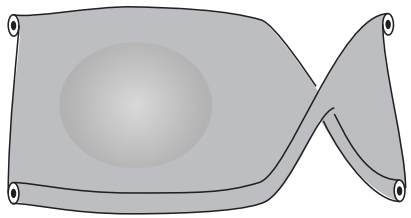
Generalized decomposition along spheres and spanning surfaces

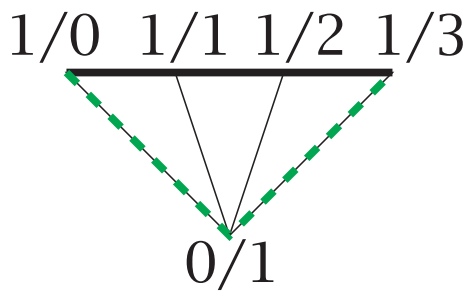
- With suitable choices, we obtain persistence, and every component of $N(\Sigma)^c$ is a taut sutured manifold.

Example: Branched Surfaces in the Complement of $T(1/3)$



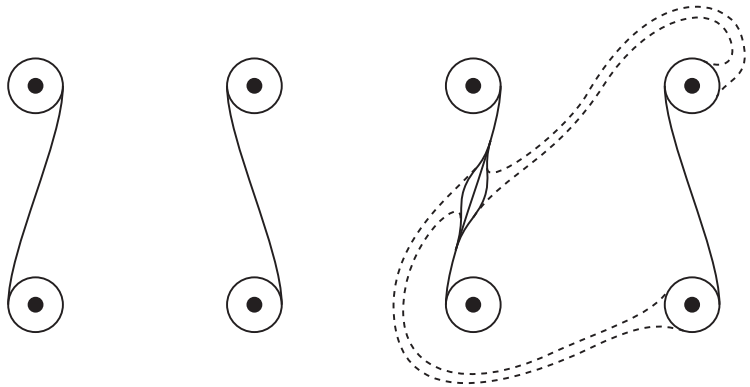


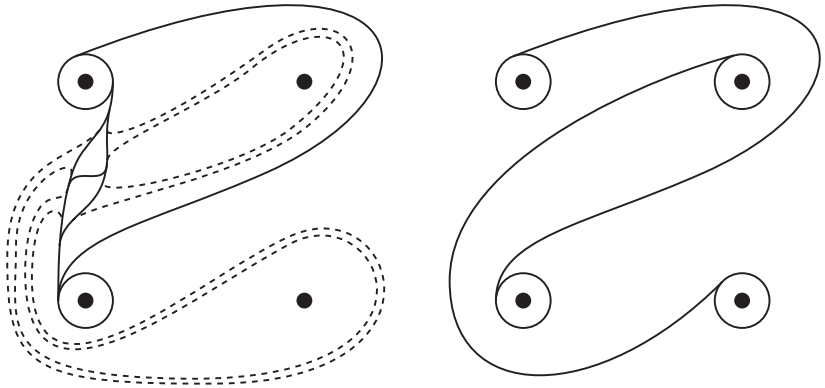


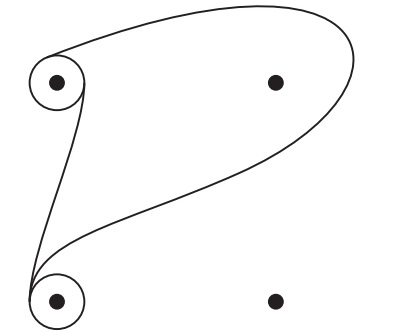


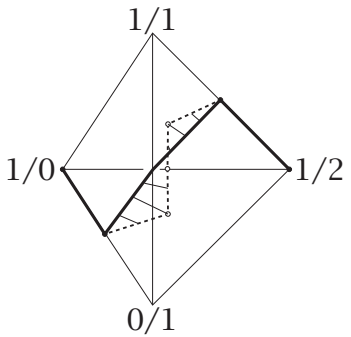
- Branched surfaces $\leftrightarrow$ paths in the Farey diagram.
- From outside the tangle, we see a twisted band.

Channel Branched Surface: Level Set Sequence

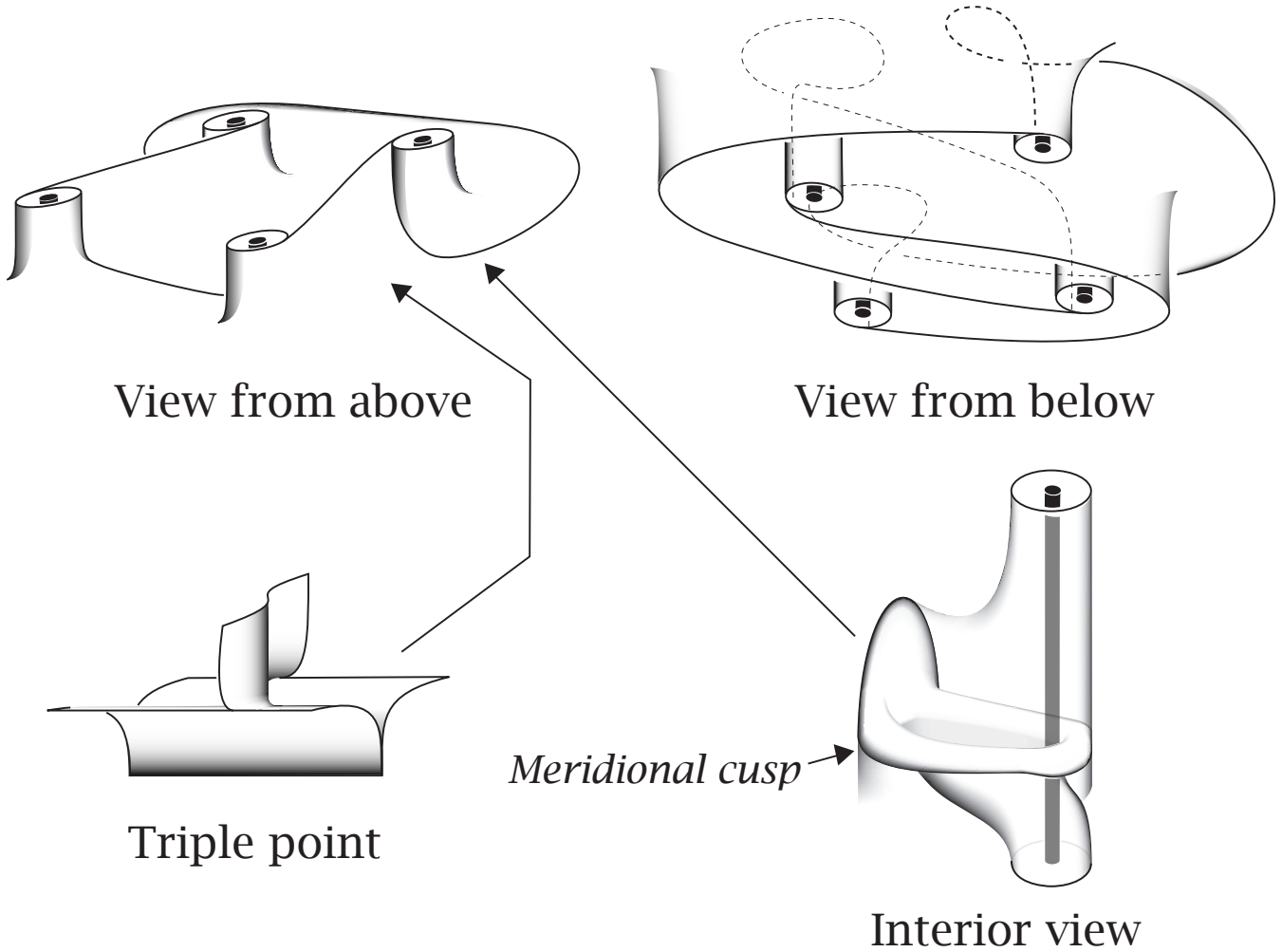








Channel Branched Surface



Combining Rational Tangles: the Enveloping Surface

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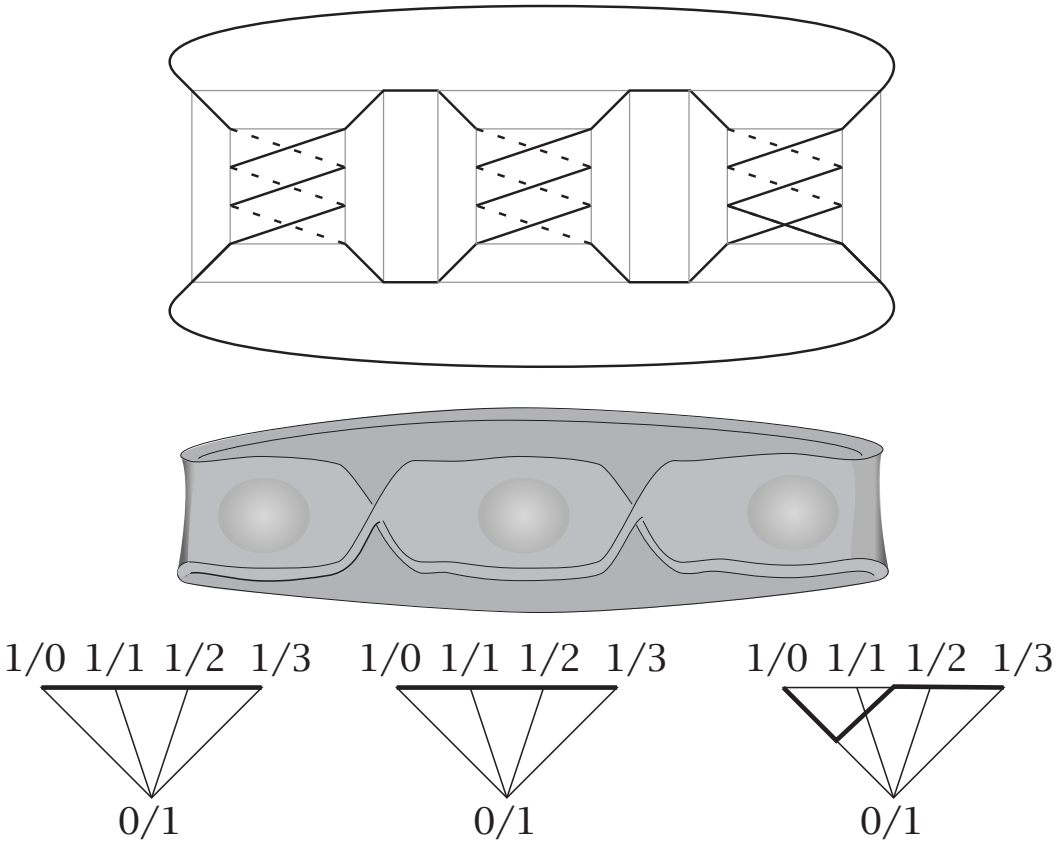
Joint work with Rachel Roberts



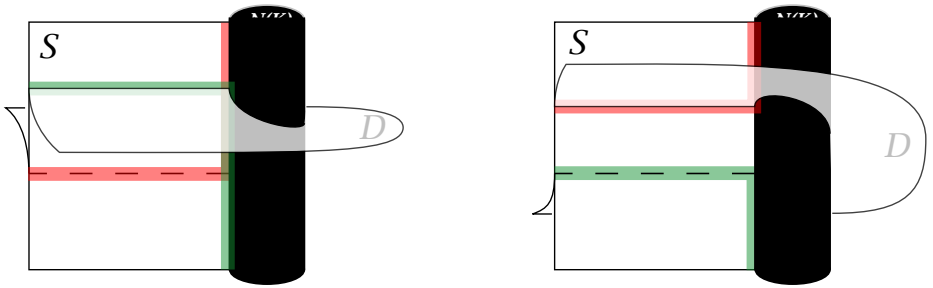
The $(3,3,3)$ pretzel knot, $K(1/3,1/3,1/3)$, is persistently foliar!

Application of Method 1

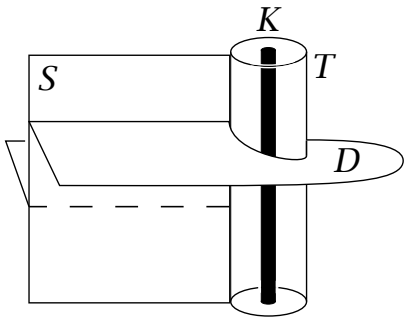
- Method 1 works well for Montesinos knots, since they decompose into rational tangles.
- Method 1 shows all Montesinos knots to be persistently foliar except for some "small" pretzel knots.

Method 2: Decomposition of a Spanning Surface

May be viewed as a generalization of Gabai's theory:



Sutured manifold decomposition
of a Seifert surface



Generalized surface decomposition
of a spanning surface

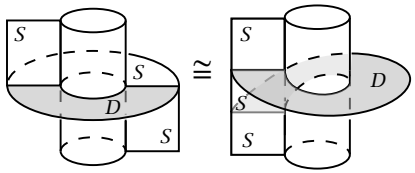
Some Differences; Application

Generalized decomposition of a spanning surface provides much greater flexibility:

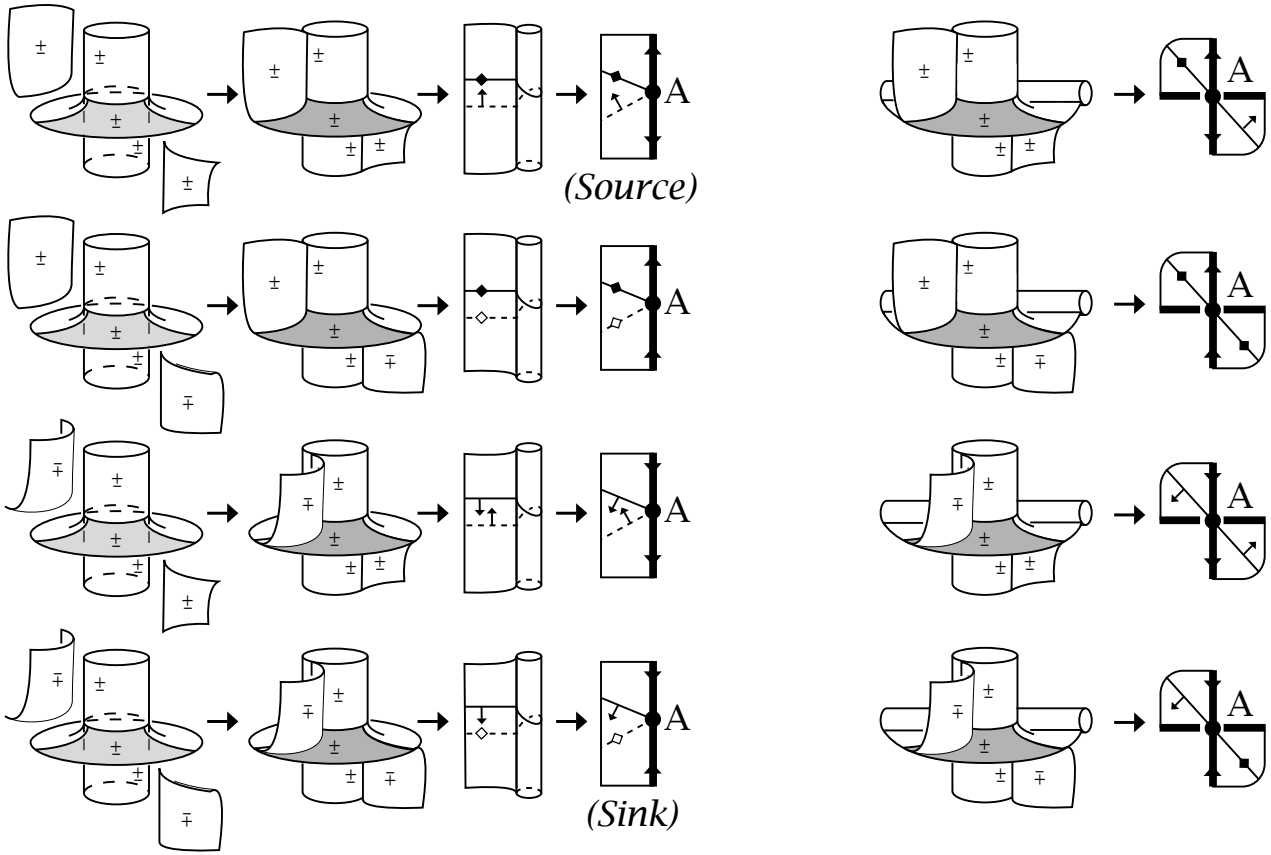
- Persistence.
- Initial spanning surface need not be orientable!
- Boundary of decomposing surface can cross over T from one side of S to the other an odd number of times!

Method 2 shows all non-torus alternating knots and all remaining pretzel knots that are not L-space knots to be persistently foliar.

Local Models and Notation Conventions: Type A



With positive twist:



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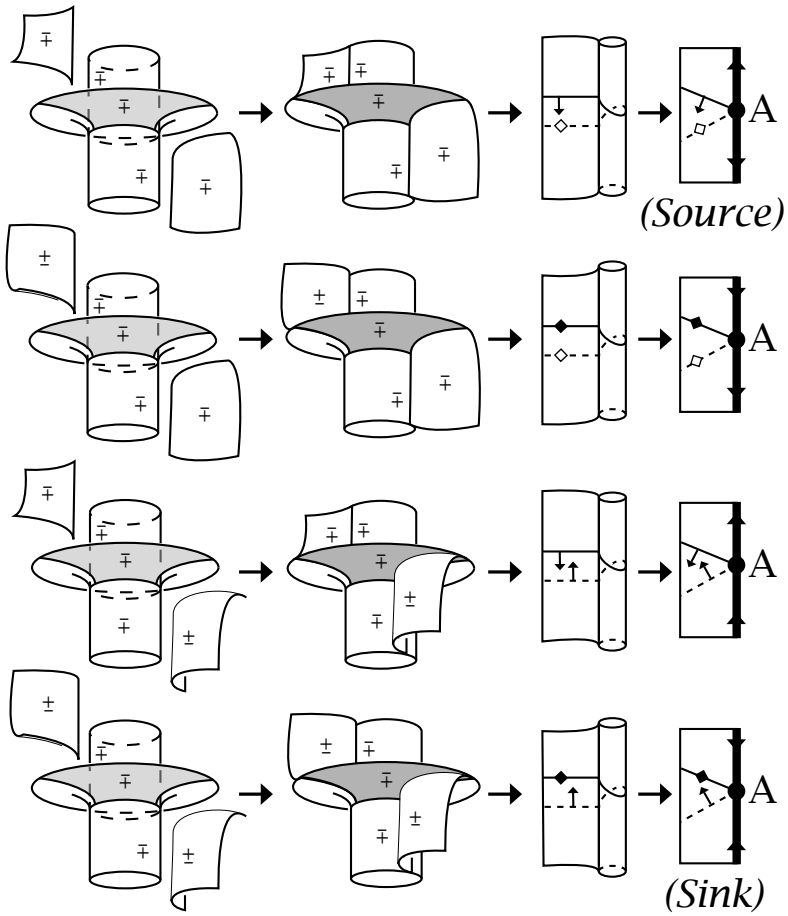
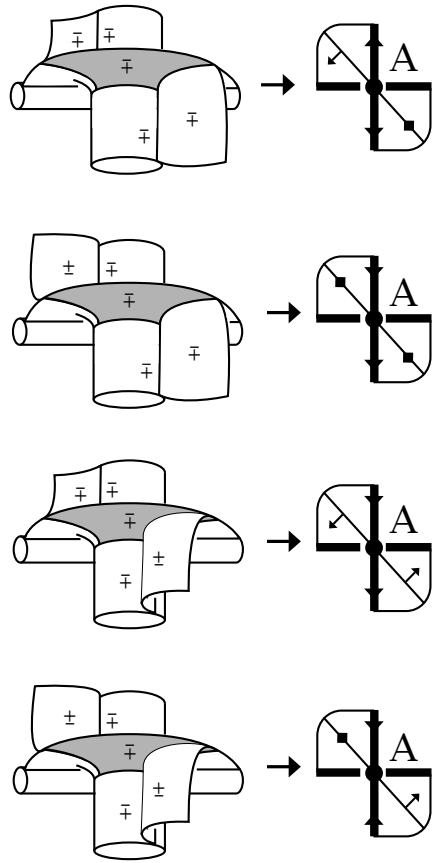
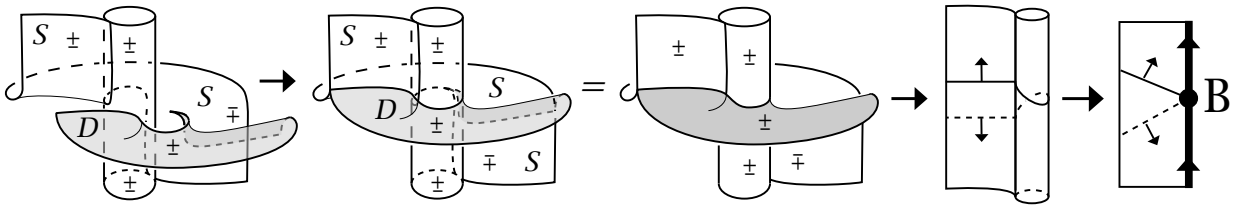
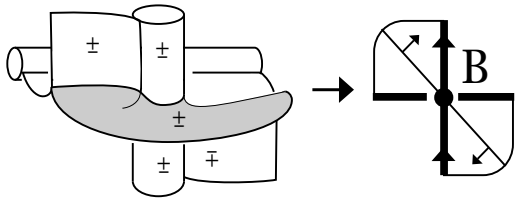


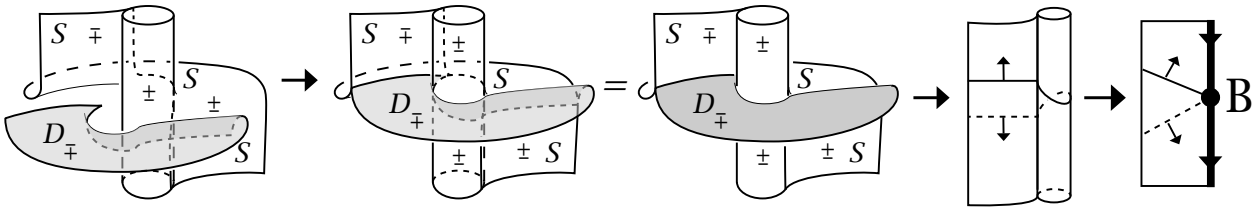


Local Models and Notation Conventions: Type B

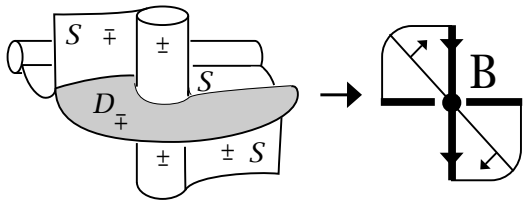


With positive twist:

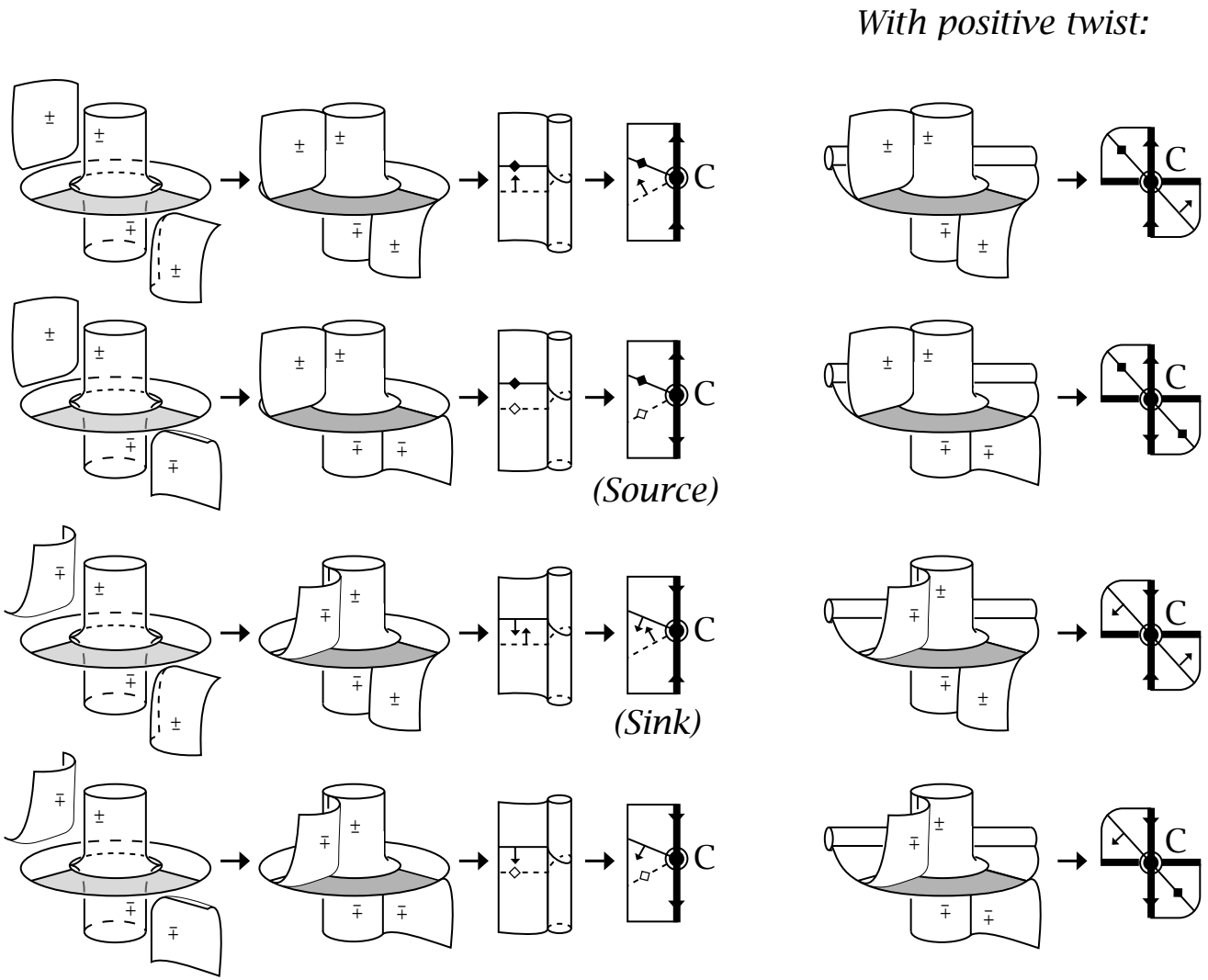




With positive twist:



Local Models and Notation Conventions: Type C



The $(-2, 5, 5)$ Pretzel Knot is Persistently Foliar

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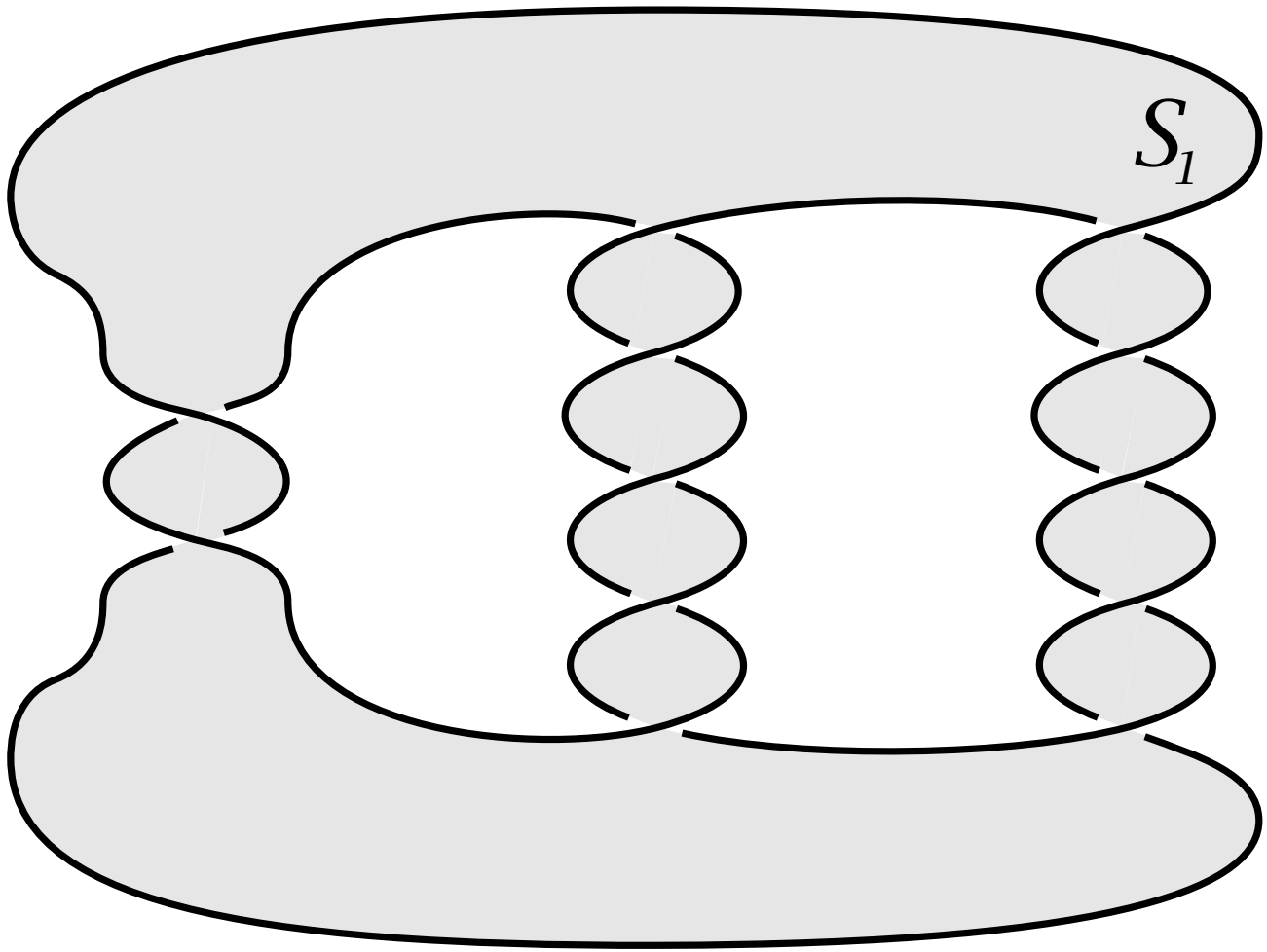

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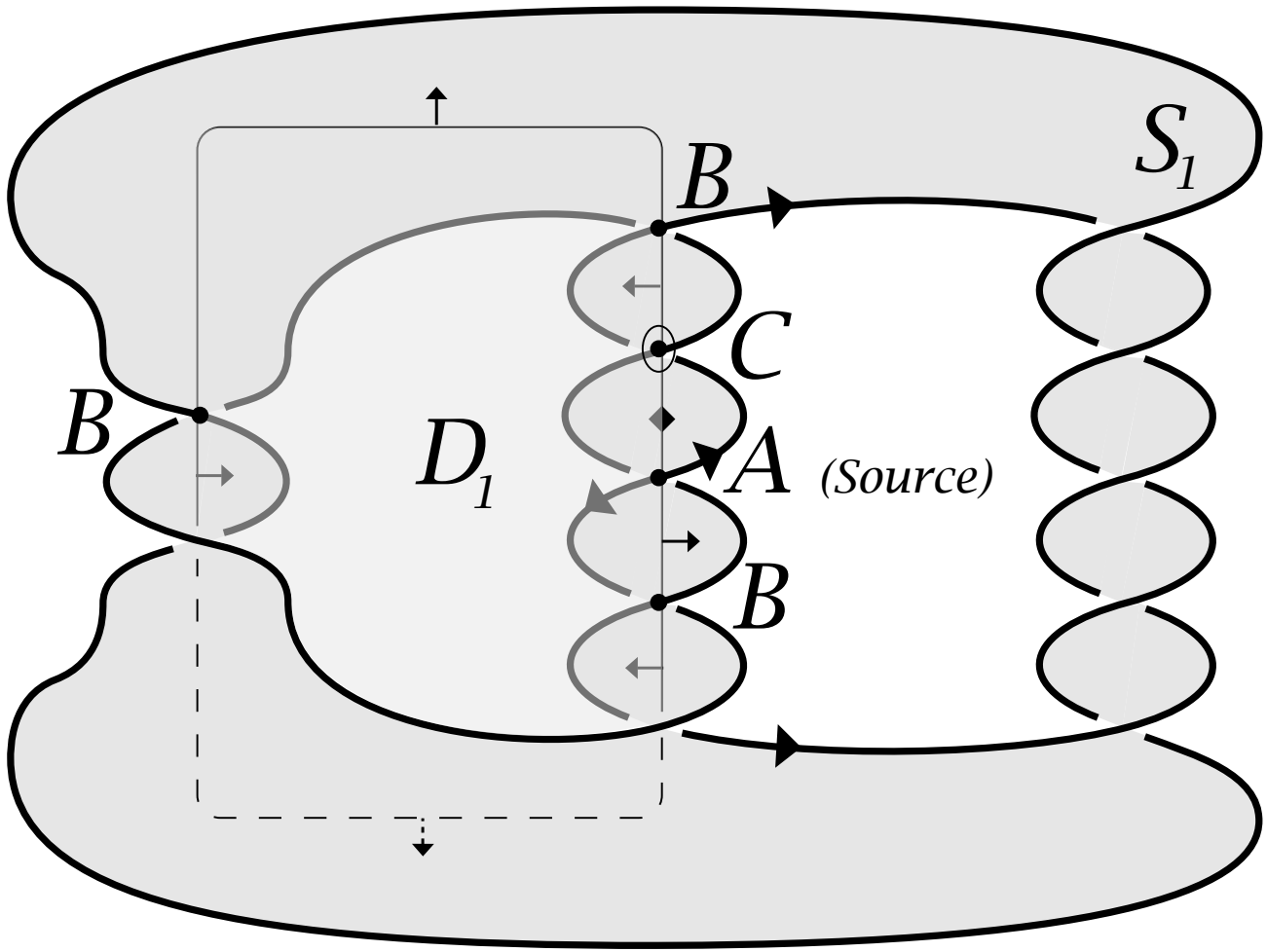

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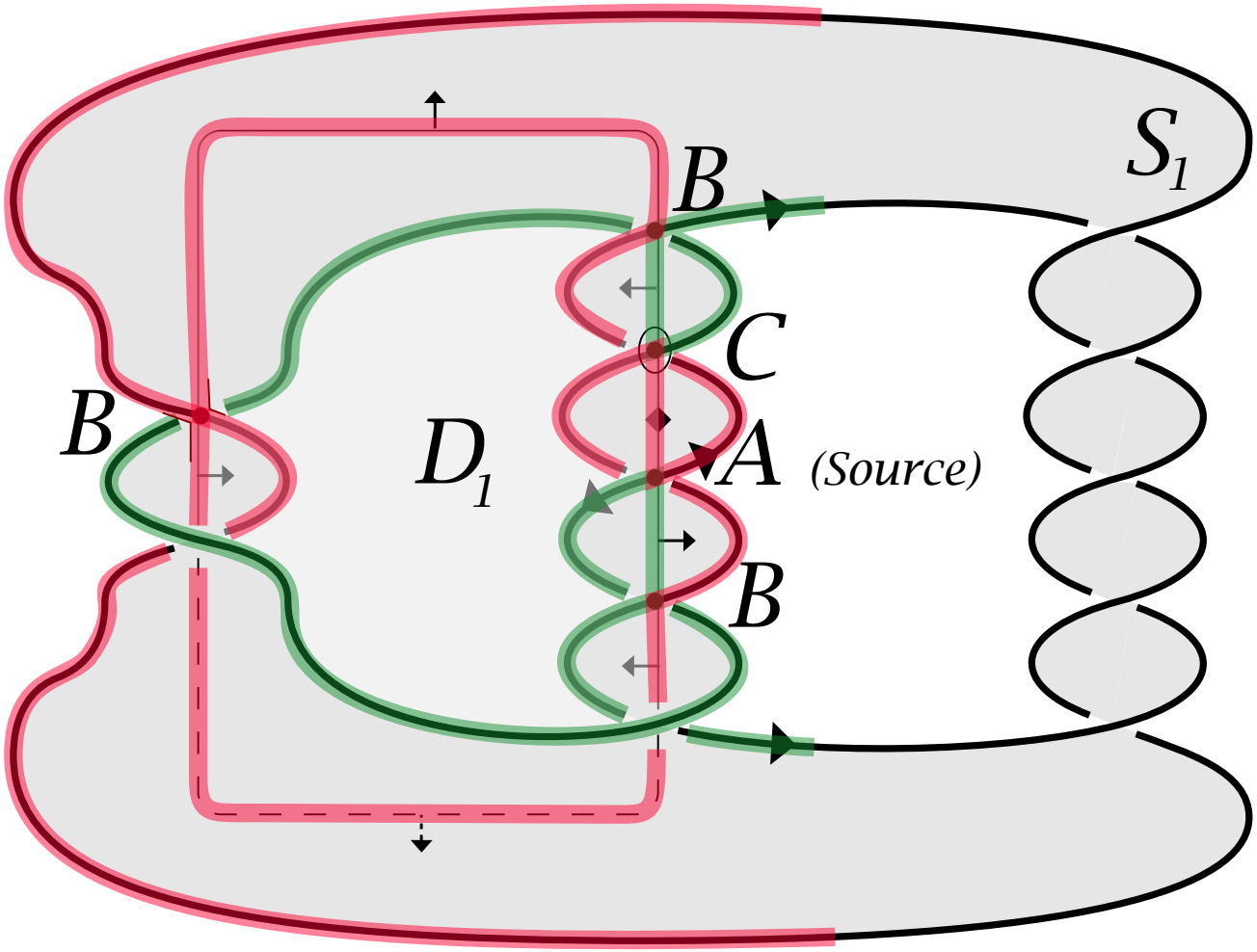

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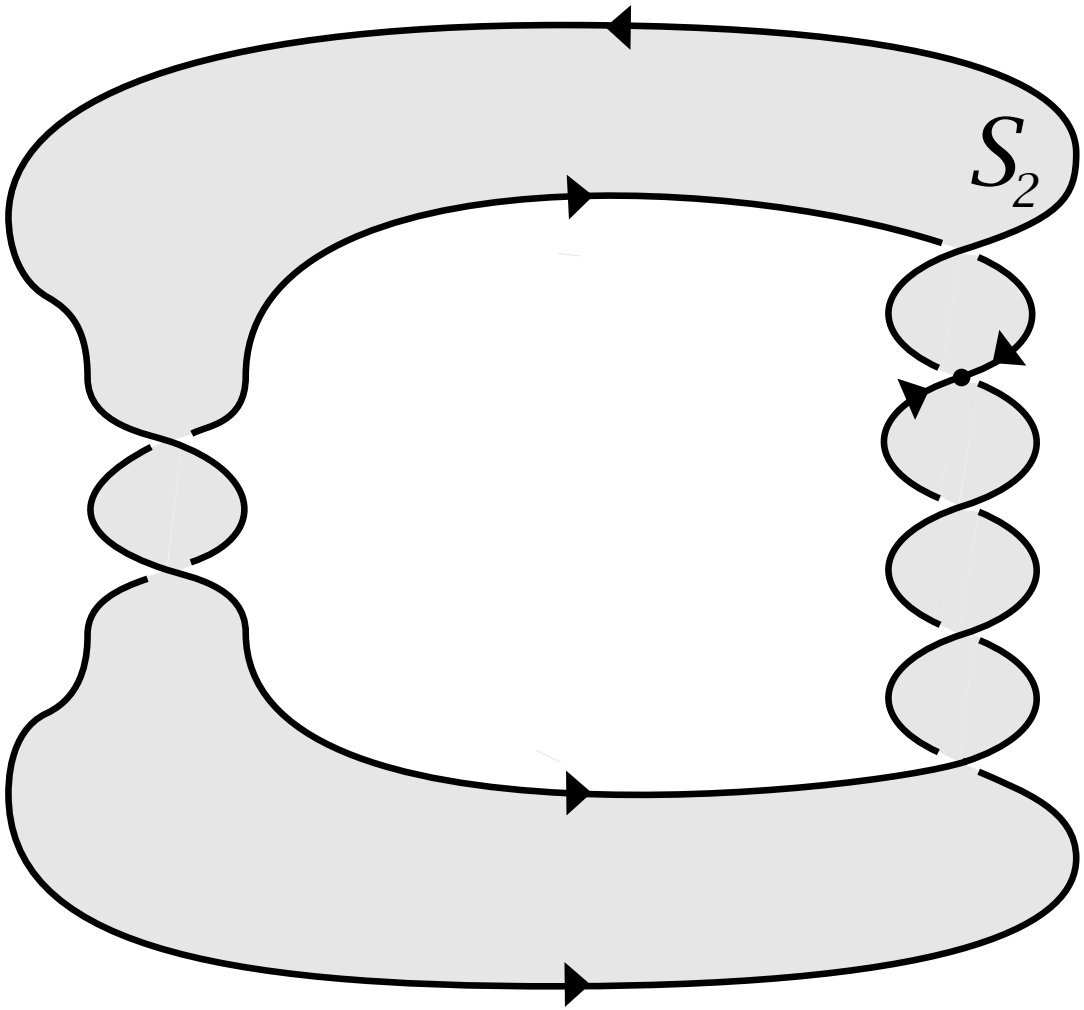

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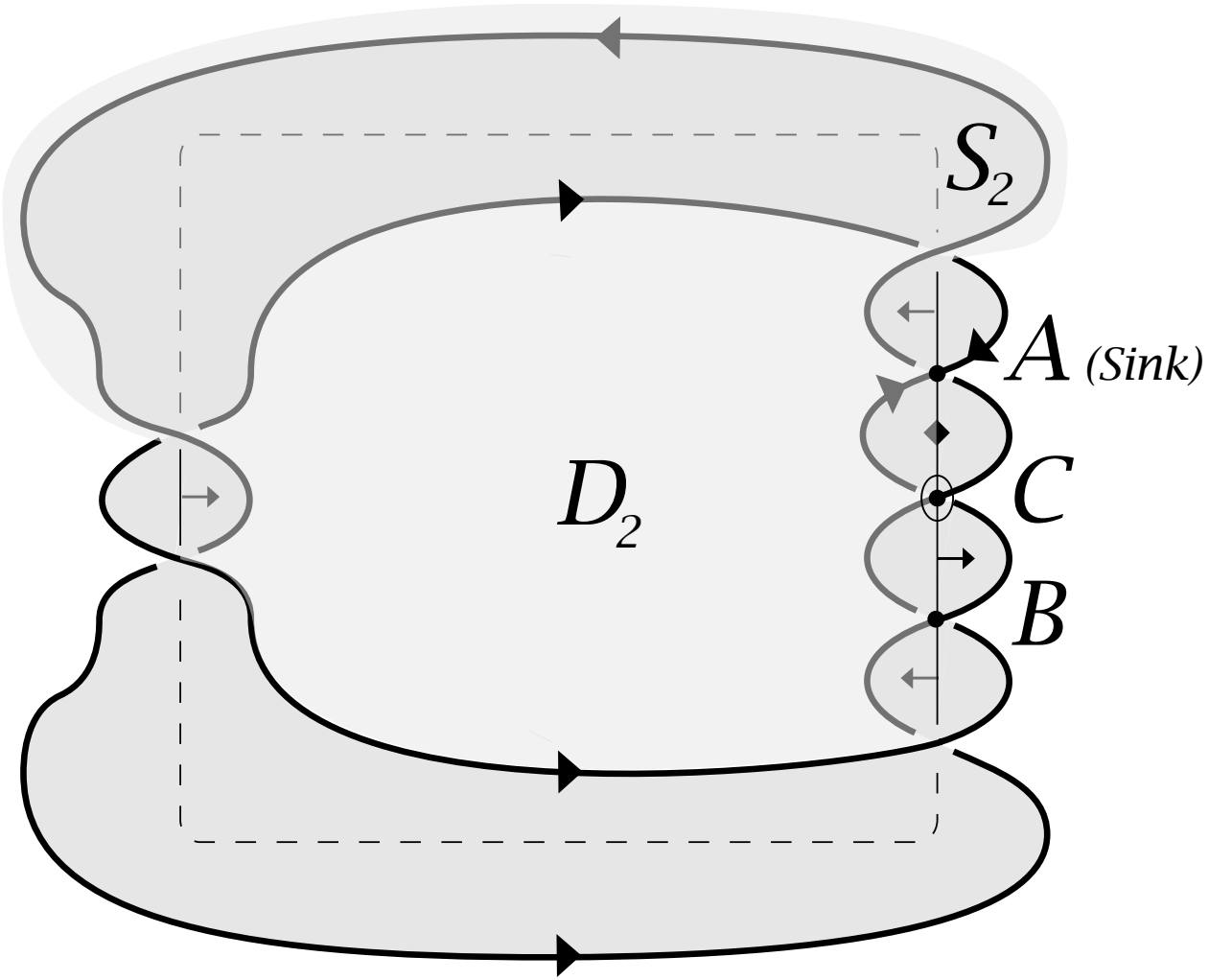

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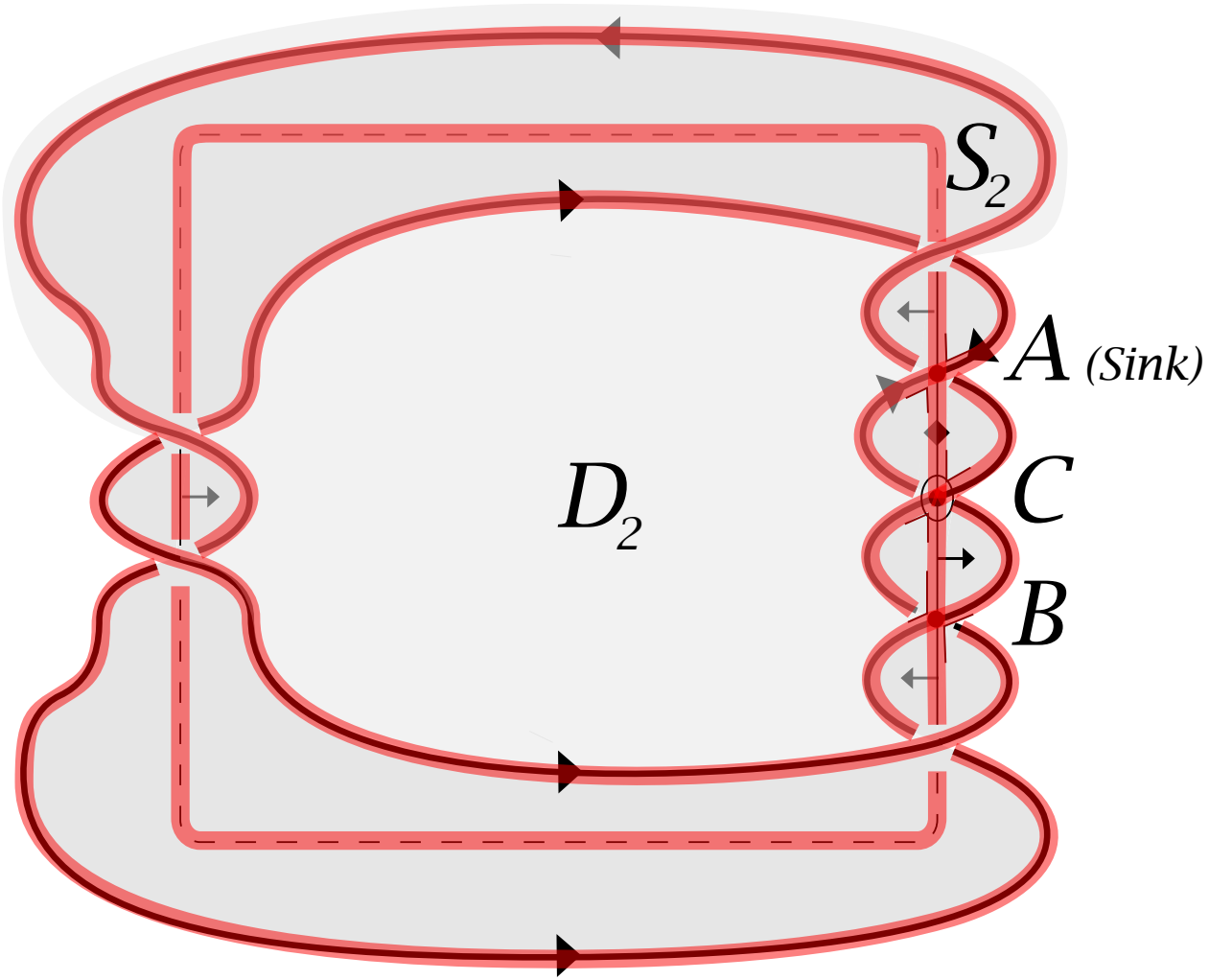

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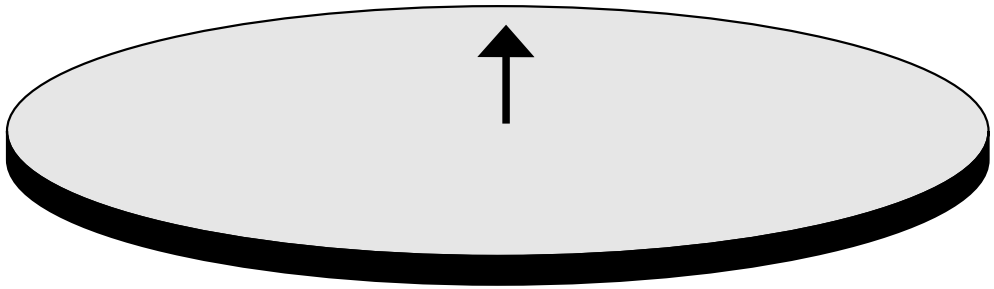
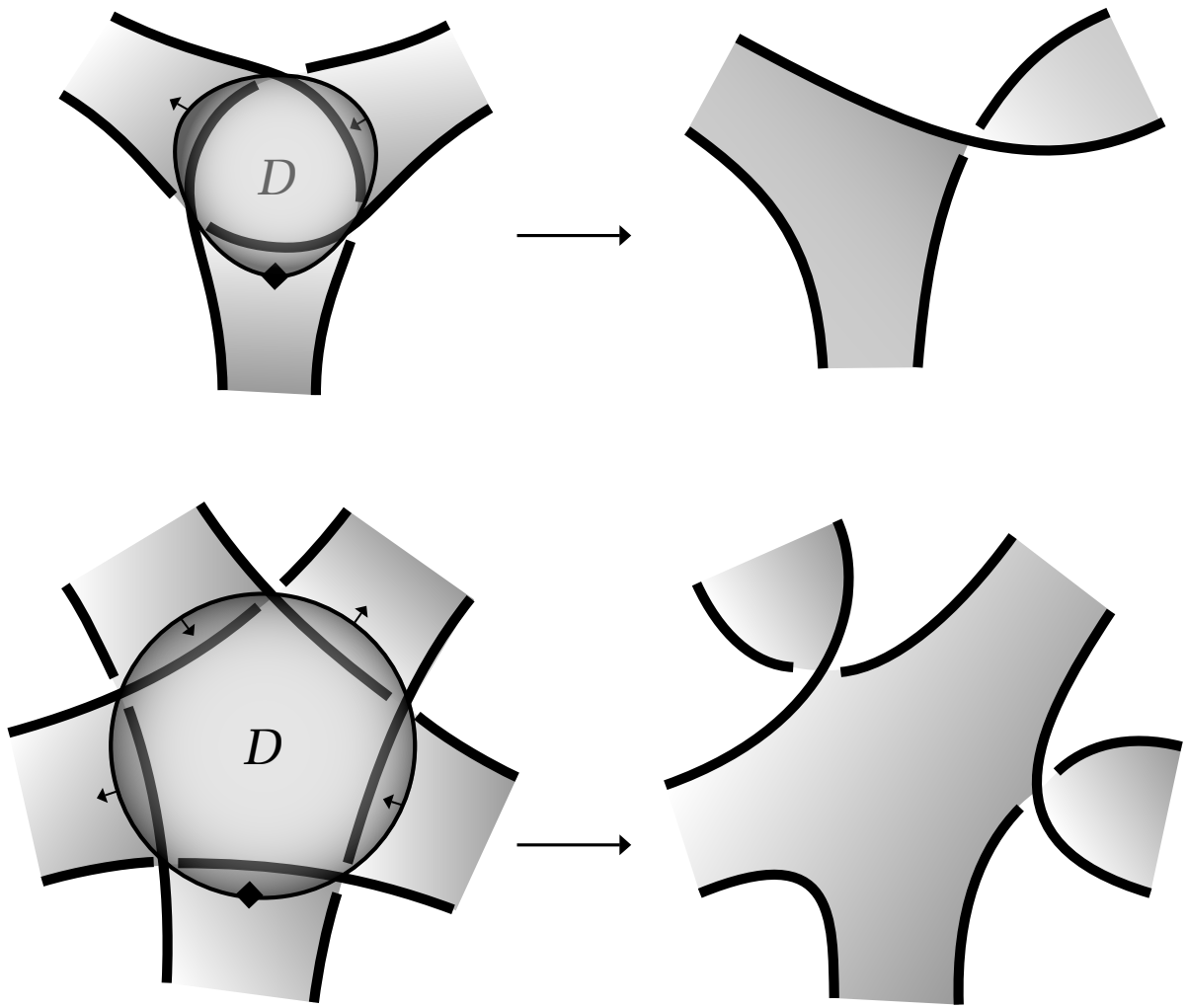


Sample Disk Decompositions in the Alternating Setting



Questions?

Thank you!