

An Introduction to the Poincaré and Geometrization Conjectures

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The Poincaré Conjecture

- ❖ Notation: Let M^n denote a compact manifold of dimension n . Let S^n denote the sphere of dimension n .
- ❖ The classical Poincaré Conjecture states that

$$\pi_1(M^3) = 0 \Rightarrow M^3 \approx S^3.$$

Category Matters (but not in dimension three)

- ❖ The equivalence that is meant is topological, that is, homeomorphism.
- ❖ In dimension three, the topological, differentiable, and polyhedral categories are equivalent.
- ❖ But in higher dimensions there are "exotic" spheres, although the topological generalization of the Poincaré Conjecture is true.

History of the classical Poincaré Conjecture

- ❖ Poincaré originally purported (1900) that any compact n -manifold with the same homology as S^n is homeomorphic to S^n .
- ❖ But four years later he himself found a three-dimensional counterexample with non-trivial finite fundamental group, the icosahedral space.
- ❖ J.H.C Whitehead purported to prove (1934) that every open contractible 3-manifold is homeomorphic to \mathbb{R}^3 , which would imply the Poincaré Conjecture.
- ❖ Except that he was wrong, too, and also found his own mistake, in the process discovering the Whitehead Link and Whitehead Manifold.

Higher Dimensions

- ❖ The three-dimensional Poincaré Conjecture persistently defied proof.
- ❖ Meanwhile, dramatic progress was made in higher dimensions.
- ❖ In higher dimensions, triviality of the fundamental group is insufficient to characterize the homotopy type of a sphere.
 - ◆ For example, consider $S^2 \times S^2$.
 - ◆ So the correct theorem is

If M^n has the homotopy type of S^n , then $M^n \approx S^n$.

Proofs of the Poincaré Conjecture in Higher Dimensions

- ❖ Smale (1960), $n \geq 5$, using differentiable methods, cobordism. (“What really happened on the beaches of Rio.”)
- ❖ Wallace (1961), $n \geq 5$, cobordism.
- ❖ Stallings (1960), $n \geq 7$, extended by Zeeman (1962) to $n \geq 5$, using piecewise-linear methods.
- ❖ Freedman, $n = 4$, unique methods. Classified all closed simply connected topological 4-manifolds.

Why Dimensions ≥ 5 are easier

- ❖ Relations in a presentation of the fundamental group are represented by disks.
- ❖ If the ambient manifold has dimension at least five, then two such disks may be made disjoint, and each disk may be freed of self-intersections

Geometric Structures

- ❖ A *geometric structure* is a homogeneous complete metric.
 - ◆ A geometric structure is obtained by restricting the transitions between charts of a differentiable manifold to a Lie group of transformations of a connected and simply connected manifold (the *model*).
 - ◆ Homogeneous means the Lie group of the model acts transitively.
 - ◆ Geometric means the point stabilizers are compact. This condition excludes affine manifolds, for example. It provides an invariant metric.
 - ◆ A closed manifold with a geometric structure is complete.
 - ◆ We also require maximal symmetry; that is, the Lie group of the model must be maximal with respect to the previous condition.

The Geometrization Conjecture

- ❖ In dimension three, there are eight model geometries.
- ❖ We will be restricting our attention to one of them, the *elliptic* model provided by isometries of the geometric sphere.
 - ◆ This model is *isotropic*: the point stabilizers are the whole rotation group $SO(3)$.
 - ◆ This model has constant positive curvature.
- ❖ The Geometrization Conjecture states that every 3-manifold may be cut along 2-spheres and 2-tori into pieces that each have a geometric structure based on one of the eight models.
- ❖ Our (more modest) goal is to prove that any 3-manifold with finite fundamental group is elliptic.

The Geometrization Conjecture, Continued

- ❖ The Geometrization Conjecture, and much of the seminal mathematics underlying it is due to Bill Thurston.
- ❖ The Geometrization Conjecture provides a context within which to understand the Poincaré Conjecture.
- ❖ The eight model geometries are \mathbb{H}^3 , \mathbb{R}^3 , S^3 , $\mathbb{H}^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}$, $\widetilde{SL_2(\mathbb{R})}$, Nil, and Solv.
- ❖ We will now explore some of these through examples.