16. Consider the Real Affine Plane (that is what the text means by the “usual Euclidean model”), which is a model of incidence geometry. Define $A \ast B \ast C$ to mean that $B$ is the midpoint of $AC$. (We make the usual interpretations of distance and congruence in the Real Affine Plane.) We need to check BAI and BAII hold and find a counterexample to BAIII.

BAI: If $B$ is the midpoint of $AC$, then $B$ is on line $AC$ by definition, and $B$ is also the midpoint of $CA$, which is the same segment. Hence, in our interpretation, if $A \ast B \ast C$, then points $A$, $B$, and $C$ are collinear and $C \ast B \ast A$.

BAII: Given any two distinct points $B$ and $D$, let $C$ be the midpoint of $BD$. Point $C$ is simply the point whose coordinates are the average of those for $B$ and $D$. Let $A$ be the point on the ray opposite $\overrightarrow{BD}$ whose distance from $B$ is equal to the distance from $B$ to $D$. Point $A$ is simply the point whose respective coordinates have the same difference from those of $B$ as the coordinates of $B$ have from those of $D$. (Alternatively, you can solve for the coordinates of $A$ using the Pythagorean distance formula and the relationship between $x$ and $y$ given by line $\overrightarrow{BD}$.) Similarly, a point $E$ exists on the ray opposite $\overrightarrow{DB}$ whose distance from $D$ is equal to the distance from $D$ to $B$. By construction, $C$ is the midpoint of $BD$, $B$ is the midpoint of $AC$, and $D$ is the midpoint of $CE$; hence, in our interpretation, $A \ast B \ast C$, $B \ast C \ast D$, and $C \ast D \ast E$.

To show that BAIII fails, let $A$ and $B$ be any two points, and let $C$ be the point on the ray opposite $\overrightarrow{BA}$ whose distance from $B$ is twice the distance from $B$ to $A$. Then, since none of these points is the midpoint of the segment whose endpoints are the other two, none is between the other two according to our interpretation. (This is one of many ways to create a counterexample.)

17. Following the book’s hint, we alter the betweenness relation for three collinear points $A$, $B$, and $P$ in the Real Affine Plane. Whereas in the Real Affine Plane $P$ is between $A$ and $B$, in our model point $A$ is declared to be between the others. Both $P \ast A \ast B$ and $B \ast A \ast P$ are declared to hold. For all other triples of points (including those involving any of the points $A$, $B$, or $P$), the betweenness relation is the same as in the Real Affine Plane. (The incidence relation also remains the same.) The incidence axioms will, of course, be unaffected by this alteration. Let us see how the betweenness axioms are affected. Since we know they hold for the Real Affine Plane, only situations involving our special points will have to be checked.

BAI: The only triple of points for which anything might change is $P$, $A$, and $B$. By definition, both $P \ast A \ast B$ and $B \ast A \ast P$ (and only these) hold, and it was assumed that these three points are distinct and collinear, because in the Real Affine Plane we had $A \ast P \ast B$. So BA1 holds for all triples of points.

BAII: We consider a pair of points $P$ and $R$ (I am carefully avoiding the labels of our special points) and note that, in the Real Affine Plane, we can choose points $O$, $Q$, and $S$ as required by the axiom so that none of them coincides with $A$, $B$, or $P$. (There are infinitely many points that satisfy each requirement.) The betweenness relations $O \ast P \ast Q$, $P \ast Q \ast R$, and $Q \ast R \ast S$ then remain the same in our model, since at least one of the points in each triple involved is not one of the special ones (even if both $P$ and $R$ are among the special ones).
BAIII: This axiom clearly holds, since for any three collinear points other than A, B, and P the usual betweenness relation holds (thus exactly one is between the other two), and for A, B, and P, A, and only A, is defined to be between the other two.

To show that BAIV fails, let C be a point such that C * A * B. We show the P (our special point P, not an arbitrary point) does not belong to either ray ĀB or ĀC. Clearly P ∉ ĀB, since P * A * B in our interpretation. On the other hand, we know that in the usual interpretation P is between A and B, so it cannot be on ray ĀC. (The normal betweenness relation applies to A, P, and C, so we know A * C * P.)

18. For the argument needed to show a half-plane is convex, see the solution to Exercise 13. To show that the interior of an angle or triangle is convex, apply the following useful (and easy to prove) general theorem, using the fact that the interior of an angle is the intersection of two half-planes and the interior of a triangle is the intersection of two angle interiors. This is a nice example of how you can simplify your work by finding a general pattern. Often, proving this general pattern just involves simple use of the definitions - once you notice what the pattern is!

**Theorem.** The intersection of two convex sets is convex. (By repeated application - that is, more formally, mathematical induction - it follows that the intersection of any finite number of convex sets is convex.)

**Proof.** Let S and T be convex sets. Let A and B be points of S ∩ T. Since A and B belong to S (definition of intersection) and S is convex, AB ⊆ S. Similarly, AB ⊆ T. Hence, AB ⊆ S ∩ T (definition of intersection). Thus, by definition, S ∩ T is convex.

To prove that the exterior of a triangle is not convex, we show the existence of two points A and B in the exterior of a triangle such that the segment AB is not contained in the exterior of the triangle. Given △PQR, let S be any point between P and Q (some such point exists by BAII), and let T be any point between Q and R. By BAII again, there exist points A, B, and C such that A * S * C * T * B. By Proposition 3.7, A is not in the interior of ∠Q, so A is in the exterior of the triangle. Similarly, B is in the exterior of the triangle. By two applications of Proposition 3.3 (or, alternatively, Proposition 3.5; supply the details in either case) we find that A * C * B, so C ∈ AB. However, it is easy to show (details left to you) that C is in the interior of the triangle, not the exterior.

A triangle (considered as the union of its three sides) is not a convex set because, by a similar argument, a segment with endpoints on two different sides of the triangle contains points that are in the interior of the triangle and not in any side.

21. (along with 22 and 23): The proof of Proposition 3.13 (Segment Ordering)

(a) Given two segments AB and CD, we must show that one and only one of the following holds: AB ≃ CD, AB < CD, or AB > CD. Please keep in mind that we are not talking about “length”. No notion of length has been formally introduced. We must use the definition of AB < CD (and the properties of the undefined term “congruent”).

I will give two proofs as examples of how one might organize the arguments. The first proves at the same time that one and only one property holds, whereas the second separates
the proof that at least one property holds from the proof that a unique one holds. Which approach to use is a matter of preference.

Proof 1: By CA1, on ray \( CD \) there is a unique point \( E \) such that \( AB \sim CE \). There are three mutually exclusive possibilities, exactly one of which must hold (here the definition of ray and BAIII are implicit): \( C \ast E \ast D, D = E, \) or \( C \ast D \ast E \). We prove that \( C \ast E \ast D \iff AB < CD, E = D \iff AB \simeq CD, \) and \( C \ast D \ast E \iff AB > CD \). (In showing the equivalence of each ordering property to a distinct betweenness property, we prove at once that one and only one ordering holds, since one and only one of the betweenness properties holds.)

If \( C \ast E \ast D \), then \( AB < CD \) by definition. Conversely, if \( AB < CD \), then by definition there is a point \( F \) between \( C \) and \( D \) such that \( AB \simeq CD \), and \( E \) is the unique point on \( CD \) with this property (CA1).

If \( E = D \), then \( AB \simeq CD \) because every segment is congruent to itself, and congruence is transitive (CA2). Conversely, if \( AB \simeq CD \), then \( E = D \) by the uniqueness part of CA1.

If \( C \ast D \ast E \), then by Proposition 3.12 there is a point \( F \) between \( A \) and \( B \) such that \( CD \simeq AF \). Thus \( AB > CD \) by definition. Conversely, assume \( AB > CD \). By definition there is a point \( F \) between \( A \) and \( B \) such that \( AF \simeq CD \). Suppose \( E = D \). Then \( AB \simeq CD \) and \( AF \simeq CD \), contradicting CA1. If \( C \ast E \ast D \), then by Proposition 3.12 there is a point \( G \) between \( A \) and \( F \) such that \( AG \simeq CE \). By Proposition 3.3, \( G \) is between \( A \) and \( B \); in particular, \( G \neq B \). But both \( AG \) and \( AB \) are congruent to \( CE \), again contradicting CA1. Therefore, the remaining relation must hold: \( C \ast D \ast E \).

Proof 2: First we prove that one of the properties \( AB < CD, AB \simeq CD, \) or \( AB > CD \) must hold. We introduce point \( E \) as before and use the three possibilities for how \( E \) relates to \( C \) and \( D \) to divide our argument into cases.

Case 1. \( C \ast E \ast D \). In this case \( AB \simeq CD \) by definition.

Case 2. \( E = D \). In this case \( AB \simeq CD \) by CA2.

Case 3. \( C \ast D \ast E \). By Proposition 3.12, there is a point \( F \) between \( A \) and \( B \) such that \( AF \simeq CD \). Thus, by definition, \( AB > CD \). (Note: It might be worth formulating the result of this little argument as a lemma for future use: if there is a point \( E \) such that \( C \ast D \ast E \) and \( AB \simeq CE \), then \( AB > CD \). The converse is also true. I suggest you prove it as an exercise.)

Next we prove by contradiction that only one property holds. Suppose that both \( AB < CD \) and \( AB \simeq CD \). From the first statement it follows by definition that there is a point \( F \) between \( C \) and \( D \) such that \( AB \simeq CF \). But together with the second statement this contradicts CA1. Exchanging the roles of the two segments in this argument shows that to have \( AB \simeq CD \) and \( AB > CD \) is also impossible. Finally, suppose \( AB < CD \) and \( AB > CD \). I leave it as an exercise to apply Proposition 3.12 and again deduce a contradiction to CA1.

(b) Assume \( AB < CD \) and \( CD \simeq EF \). By definition, there is a point \( G \) between \( C \) and \( D \) such that \( AB \simeq CG \). By Proposition 3.12, there is a point \( H \) between \( E \) and \( F \) such that \( CG \simeq EH \). By the transitivity of congruence (CA2), \( AB \simeq EH \), so by definition \( AB < EF \).

(c) Assume \( AB > CD \) and \( CD \simeq EF \). By definition there is a point \( G \) between \( A \) and \( B \) such that \( CD \simeq AG \). By the transitivity of congruence, \( EF \simeq AG \), so \( AB > EF \) by definition.

(d) Assume \( AB < CD \) and \( CD < EF \). By definition there is a point \( G \) between \( C \) and \( D \) such that \( AB \simeq CG \). Similarly there is a point \( H \) between \( E \) and \( F \) such that \( CD \simeq EH \). By proposition 3.12, there is a point \( I \) between \( E \) and \( H \) such that \( CG \simeq EI \). By CA2, \( AB \simeq EI \). By Proposition 3.3, \( E \ast I \ast F \). So by definition \( AB < EF \).
25. (a) Given a pair of vertical angles, each is supplementary to a common angle. This common angle is congruent to itself, so by Proposition 3.14 the vertical angles are congruent.

(b) It is important to understand two things. First, Proposition 3.15(b) is different from Euclid’s Fourth Postulate (Proposition 3.23 in our system). The hypothesis of Euclid’s Fourth Postulate is that each of two angles is a right angle. The conclusion is that the angles are congruent. In Proposition 3.15(b) we are given one right angle and another angle that is congruent to it. The conclusion is that the second angle is a right angle. Second, although it may seem that these statements “obviously should be true,” they must be proven. Given that \( \angle 1 \) is a right angle and \( \angle 2 \simeq \angle 1 \), let \( \angle 3 \) be the supplement of \( \angle 1 \) and \( \angle 4 \) the supplement of \( \angle 2 \). By Proposition 3.14, \( \angle 3 \simeq \angle 4 \). By the definition of right angle, \( \angle 3 \simeq \angle 1 \). By CA5, \( \angle 1 \simeq \angle 4 \). Thus, by definition, \( \angle 2 \) is a right angle.

Some important subtleties regarding the proofs of Propositions 3.20 and 3.21.

As we saw in class, an important difficulty arises in the proof of Proposition 3.20 when we need to show that one ray is between two other rays. Recall that, by hypothesis, \( \overrightarrow{BG} \) is between \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \), \( \overrightarrow{EH} \) is between \( \overrightarrow{ED} \) and \( \overrightarrow{EF} \), \( \angle CBG \simeq \angle FEH \), and \( \angle ABC \simeq \angle DEF \). (See the figure below; I’m using the text’s labeling.) Using CA4, we introduced a ray \( \overrightarrow{EI} \) on the opposite side of line \( \overrightarrow{EF} \) from \( F \) such that \( \angle HEI \simeq \angle GBA \). Our intention was to prove, using angle addition and the uniqueness part of CA4, that \( \overrightarrow{EI} = \overrightarrow{ED} \), concluding that \( \angle GBA \simeq \angle HED \) as desired.

The problem is: how do we prove that \( \overrightarrow{EH} \) is between \( \overrightarrow{EI} \) and \( \overrightarrow{EF} \), the necessary hypothesis for angle addition? How do we even prove that \( \overrightarrow{EI} \) is on the same side of line \( \overrightarrow{EF} \) as \( \overrightarrow{ED} \), the necessary hypothesis for CA4? (See the figures below.)
In fact there is no way to prove these things if all we know is what side of line $\overrightarrow{EH}$ ray $\overrightarrow{EI}$ is on. Betweenness for coterminal rays is more complicated than betweenness for collinear points. In the case of collinear points, if $A$ and $C$ are on opposite rays emanating from $B$, then $A \ast B \ast C$. However, if rays $\overrightarrow{PA}$ and $\overrightarrow{PC}$ are on opposite sides of line $\overrightarrow{PB}$, it does not follow that $\overrightarrow{PB}$ is between $\overrightarrow{PA}$ and $\overrightarrow{PC}$. (See the figure below. Remember, the concept of a side of a ray does not make sense! Make sure you understand why not.) Furthermore, given three (distinct) coterminal rays, one is not necessarily between the other two; closing this gap in the proof of angle ordering will require a little refinement.

![Diagram](image.png)

The solution to the problem in the proof of angle subtraction is to prove the analogue of Proposition 3.12, as stated in Exercise 30, first. (That is, we will use the result of Exercise 30 in our solution to Exercise 29.) Exercise 30 is not hard to do following the hints given. (One might wonder, given this situation, if Proposition 3.12 can be proven before Proposition 3.11. Indeed, if you examine the proof you will see that it can.) We will also see the value of Proposition 3.9(c).

Returning to the proof of angle subtraction, let $J$ and $K$ be points such that $J \ast A \ast C$ and $K \ast E \ast F$ (BA2). Ray $\overrightarrow{BA}$ is between $\overrightarrow{BG}$ and $\overrightarrow{BJ}$, by Proposition 3.9(c) and the definition of between for rays. By the analogue of Prop. 3.12 (result of Exercise 30), there is a ray $\overrightarrow{EI}$ between $\overrightarrow{EH}$ and $\overrightarrow{EK}$ such that $\angle HEI \simeq \angle GBA$. Again using Proposition 3.9(c), we obtain that $\overrightarrow{EH}$ is between $\overrightarrow{EI}$ and $\overrightarrow{EF}$!

The rest of the proof is analogous to the proof of segment subtraction. I leave it to you to show why the hypothesis of CA4 applies at the necessary step.

For the proof of angle ordering, we need the following result similar to the third problem on the second exam:

**Lemma.** Given two distinct rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ on the same side of line $\overrightarrow{PC}$, then either $\overrightarrow{PA} \ast \overrightarrow{PB} \ast \overrightarrow{PC}$ or $\overrightarrow{PB} \ast \overrightarrow{PA} \ast \overrightarrow{PC}$.

**Proof.** It follows from the hypothesis that no two of the rays $\overrightarrow{PA}$, $\overrightarrow{PB}$, and $\overrightarrow{PC}$ are opposite to each other. (Why?) Therefore, neither $B$ nor $C$ lies on $\overrightarrow{PA}$. (Why?) So $B$ and $C$ are either on the same side of $\overrightarrow{PA}$ or on opposite sides.

Case 1. They are on the same side. Then $\overrightarrow{PA} \ast \overrightarrow{PB} \ast \overrightarrow{PC}$. (Why?)

Case 2. They are on opposite sides. Then $BC$ intersects $\overrightarrow{PA}$ at a point $A'$ between $B$ and $C$.

Claim: $A'$ is on the same side of $\overrightarrow{PC}$ as $A$. The key to proving the claim is the observation that $C$ is the only point of intersection of lines $\overrightarrow{BC}$ and $\overrightarrow{PC}$ (Prop. 2.1). Suppose that $A'$ and $A$ are on opposite sides of $\overrightarrow{PC}$. Then $A'$ and $B$ are on opposite sides of $\overrightarrow{PC}$ by the corollary to BA4. Therefore it must be the case that $A' \ast C \ast B$. But $A'$ is between $B$ and $C$, contradicting BA3.

Thus $\overrightarrow{PA'} = \overrightarrow{PA}$. (Why?) By Proposition 3.7 (and the definition of between for rays), $\overrightarrow{PB} \ast \overrightarrow{PA} \ast \overrightarrow{PC}$. \qed