MAT 5335: Problems on Spherical Geometry and Stereographic Projection

Due Thursday, September 11

Explain all answers as completely as possible in essay form! Type or write neatly; I encourage you to type them using Tex. Illustrations are welcome and encouraged!

1 Spherical Geometry (2-dimensional)

All questions refer to a round 2-dimensional sphere of radius 1, with the geometry it inherits from its embedding in $\mathbb{R}^3$ as the solution set of $x_1^2 + x_2^2 + x_3^2 = 1$.

1. What is the area (that is, surface area) of the sphere?

2. What is the circumference of a circle radius $r$, as a function of $r$? (Remember that both the radius and circumference are measured on the sphere; a radius is an arc.) Compare this to the circumference of a circle in the plane.

3. What is the area enclosed by a circle of radius $r$, as a function of $r$? Compare this to the area enclosed by a circle in the plane.

4. Explain why any two (distinct) great circles intersect in two antipodal points. (Hint: A great circle is the intersection in $\mathbb{R}^3$ of the 2-sphere and a plane through the origin.)

5. A sector of the sphere is defined as a connected region bounded by a pair of great circles. (Any pair of great circles bounds two sectors, which are congruent to each other.) It should be clear that the area of a sector depends only on the angle between the great circles. What is the formula for the area of a sector?

6. A remarkable and important fact is that, on a sphere, the area of a triangle depends only on the sum of its angles. What is the area formula? (Hints: Try some simple examples, such as a triangle with three right angles. If you need help, consult “A Tale of Three Circles,” by Delman and Galperin.)

7. Generalize the above two results to give a formula for the area of a polygon with $n$ sides.

2 Spherical Geometry (3-dimensional)

All questions refer to a round 3-dimensional sphere of radius 1, with the geometry it inherits from its embedding in $\mathbb{R}^4$ as the solution set of $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. (If you like complex numbers, you might sometimes find it useful to think of the sphere as the solution to $z_1^2 + z_2^2 = 1$ in $\mathbb{C}^2$. This fact gives the 3-sphere an amazing amount of structure and symmetry!)

1. What is the (3-dimensional) volume of the sphere? (That is, the volume of the sphere itself, not the 4-dimensional volume of the 4-dimensional ball it encloses.)

2. What is the area (that is, surface area) of a 2-dimensional sphere (in the 3-sphere) of radius $r$, as a function of $r$? Compare this to the surface area of sphere in flat 3-dimensional space (that is, $\mathbb{R}^3$ with the Pythagorean metric).
3. What is the volume enclosed by a 2-dimensional sphere of radius \( r \), as a function of \( r \)? Compare this to the volume of a sphere in flat 3-dimensional space.

4. A great 2-sphere is a 2-sphere (in the 3-sphere) of radius 1. Note that this means it is the intersection in \( \mathbb{R}^4 \) of the unit 3-sphere with a 3-plane through the origin. Explain why any two great 2-spheres intersect in a great circle (that is, a circle of radius 1, which is the intersection in \( \mathbb{R}^4 \) of the unit 3-sphere with a 2-plane through the origin).

5. A sector of the 3-sphere is defined as a connected region bounded by a pair of great 2-spheres. (Any pair of great 2-spheres bounds two sectors, which are congruent to each other.) It should be clear that the volume of a sector depends only on the (dihedral) angle between the great circles. What is the formula for the volume of a sector?

6.* For extra credit, find and explain area formulas for any other type of region you wish.

### 3 Stereographic Projection

1. In the stereographic projection of the 2-sphere (minus the “North Pole”) onto the plane, what does the image of the equator look like?

2. What does the image of a meridian look like? (A meridian is a great circle passing through both the North and South Poles; that is, through \((0,0,1)\) and \((0,0,-1)\).)

3. What does the image of a circle that passes through the North Pole but is not a meridian look like?

4. How do you recognize that a circle in the stereographic projection of the 2-sphere is the image of a great circle, assuming you know the image of the equator?

5. How do you recognize antipodal points on a great circle?

6. In the stereographic projection of the 3-sphere (minus the “North Pole”) onto the plane, what does the image of the equator look like? (The equator is the 2-sphere satisfying \(x_4 = 0\). The North Pole is the point \((0,0,0,1)\), or \((0,i)\) in \(\mathbb{C}^2\).)

7. What does the image of a meridional 2-sphere look like? (A meridional 2-sphere is a great sphere passing through both the North and South Poles; that is, through \((0,0,0,1)\) and \((0,0,0,-1)\), or \((0,i)\) and \((0,-i)\) in \(\mathbb{C}^2\).) What does the image of a meridional circle look like?

8. What does the image of a 2-sphere that passes through the North Pole but not the South Pole look like?

9. How do you recognize that a 2-sphere in the stereographic projection of the 3-sphere is the image of a great 2-sphere, assuming you know the image of the equator? How do you recognize the image of a great circle? How do you recognize antipodal points?