MAT 5335: Problems on Cut-and-paste diagrams, Homotopy, and Homology

Due Thursday, September 18

Explain all answers as completely as possible in essay form! Type or write neatly; I encourage you to type them using Tex. Illustrations are welcome and encouraged!

1 Cut-and-paste diagrams

1. The surface created by the gluing diagram below is called a Klein bottle (after the great geometer Felix Klein).

   (a) Explain why the Klein bottle has flat (Euclidean) geometry.
   (b) Explain why the Klein bottle is non-orientable.

2. The surface created by gluing two opposite sides of a square with opposite orientation, as with a torus, while leaving the other two sides unglued, is called an annulus. (See diagram below.) Annulus means ring; an annulus is also often called a band or cylinder, depending on context. An annulus is a surface with boundary; it has two boundary components, each of which is topologically a circle.
(a) Explain why an annulus has flat geometry and, furthermore, each of its boundary components is geodesic (that is, straight in the sense of locally being the shortest path between any two points on it).

(b) Show that removing an open disk from a sphere produces a closed disk. The closed disk has flat geometry, of course, but note that its boundary circle is not geodesic. (Why not?)

(c) Show by appropriate cutting, gluing, and deformation of diagrams that removing an open disk from a closed disk (in other words, removing two open disks from a sphere) produces an annulus. Note that deformation is necessary to produce a flat annulus with geodesic boundary, since the boundary of a disk is not geodesic.

3. The surface created by the gluing diagram below, in which a pair of opposite sides is glued with the same orientation (thus creating a twist if you do it in space), is called a Möbius band (also after a mathematician). Note that the Möbius band has only one boundary component.
(a) Explain why the Möbius band has flat geometry and, furthermore, its boundary is geodesic.
(b) Explain why the Möbius band is non-orientable.
(c) Show by appropriate cutting, gluing, and deformation of diagrams that removing an open disk from a projective plane produces a Möbius band. Note that deformation is necessary to produce a flat Möbius band because the projective plane is not flat; it has elliptic (that is, locally spherical) geometry.

4. Show that a Klein bottle is the union of two Möbius bands glued along their boundaries.

5. Show that the diagram below produces a projective plane.

2 Homotopy and homology

1. Consider the oriented loop labeled \( a \) in the projective plane below. Explain why \( a^2 = 1 \) in the group of homotopy classes of loops in the projective plane. (The base point doesn’t really matter, but if you like you can consider the loop \( a \) to begin and end at the point labeled \( P \).)
2. Consider the oriented curves labeled $m$, $l$, and $w$ ($m$ for meridian, $l$ for longitude, $w$ for “waist curve”) on the genus-two surface (that is, two-handled torus) below.

(a) Explain why $mlm^{-1}l^{-1} = w$ as an element of the group of homotopy classes of loops on the surface. (Again, the base point doesn’t really matter as long as you are consistent, but it is natural to consider the loops to begin and end at the point labeled $P$.)

(b) Explain why $w \neq 1$ in the group of homotopy classes of loops on the surface.

(c) Explain why $w = 0$ in the first homology group of the surface. (We say $w$ is “homologous to zero.”)