## MAT 5220: Topology

Name: $\qquad$

## Midterm Exam

## Due by 5 p.m. on Monday, February 25.

You are expected to work on this exam alone and to refrain from talking about the exam to anyone except the professor until the time and date when it is due. You may use your own notes and any published materials that you like. (Cite sources appropriately.)

Your signature below attests to a pledge that you have done the exam according to the above instructions. (Please attach this cover page to your solutions.)

## Signature:

Solutions must be typeset.

1. Recall that a map from a topological space $X$ to a topological space $Y$ is open if the image of each open set is open. (Formally, $f: X \rightarrow Y$ is open if, $\forall U \subset X$ such that $U$ is open, $f(U)$ is open in $Y$.)
(a) Prove that it suffices to consider basic open sets; that is, if the topology on $X$ is given by a basis $\mathcal{B}$, and $f(B)$ is open for all basic open sets $B \in \mathcal{B}$, then $f$ is an open map. (Hint: the image of a union is the union of the images.)
(b) Given a product space $X \times Y$, let $\pi_{X}$ be the projection $(x, y) \mapsto x$ onto the first coordinate, and let $\pi_{Y}$ be the projection $(x, y) \mapsto y$ onto the second coordinate. Prove that $\pi_{X}$ and $\pi_{Y}$ are open maps.
(c) Show that the composition of open maps is open. That is, if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are open maps, then $g \circ f: X \rightarrow Z$ is open.
(d) Show that the coordinate functions of an open map to a product space are open. That is, let $f: X \rightarrow Y \times Z$ be given by the equation $f(x)=$ $\left(f_{1}(x), f_{2}(x)\right)$. If $f$ is open, then so are $f_{1}: X \rightarrow Y$ and $f_{2}: X \rightarrow Z$.
2. (a) Let $X$ be a subspace of $Y$. Give a necessary and sufficient condition for the inclusion map $i: X \rightarrow Y, i(x)=x$, to be an open map. Prove that your condition is both necessary and sufficient. (Hint: don't forget that $Y$ is open in $Y$ !)
(b) Let $(X, \tau)$ and $\left(X, \tau^{\prime}\right)$ denote the same set $X$ with two topologies. Let $i$ : $X \rightarrow X$ be the identity map, defined by $i(x)=x$. Give a necessary and sufficient condition for the inclusion map $i: X \rightarrow Y, i(x)=x$, to be an open map. Prove that your condition is both necessary and sufficient.

Extra Credit! (Not difficult!) Is the converse to the statement of problem 1(d) valid? If so, prove it. If not, give a counterexample. (Hint: Consider the diagonal map from the real line into the plane defined by $f(x)=(x, x)$.)

