MAT 4860: Selected Solutions to Problems II

1.2.16 Suppose $A \subset \mathbb{Z}$ is a non-empty subset that is bounded below. Then A has a smallest elements.

Proof. By the Greatest Lower Bound Property, A has a greatest lower bound a_0. We claim that a_0 is the smallest element of A. Since a_0 is the greatest lower bound, $a_0 + 1$ is not a lower bound; thus, there is an element $a \in A$ such that $a_0 \le a < a_0 + 1$. It follows that $a - 1 < a_0$; hence, there is no integer between a_0 and a. Therefore, $a_0 = a$, and a_0 is the smallest element of A. $\square$

Remark. A similar argument shows that any non-empty set of integers that is bounded above has a largest element. (*Prove it!*)

1.4.1 To create a bijection from $(a, b]$ to $(c, d]$, for any real numbers $a < b$ and $c < d$, let $f : (a, b] \to (c, d]$ be defined by $f(x) = c + \frac{d-c}{b-a}(x - a)$. To prove f is bijective (which is pretty obvious in any case), just solve for the inverse function.

2.1.7 To show that $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} |x_n| = 0$, note that $|x_n - 0| = |x_n| = ||x_n|| = ||x_n| - 0|$. Thus, $\forall \epsilon > 0, \exists N \in \mathbb{N} : |x_n - 0| < \epsilon \Leftrightarrow \forall \epsilon > 0, \exists N \in \mathbb{N} : ||x_n| - 0| < \epsilon$.

2.1.10 The sequence $\left(\frac{n+1}{n}\right)$ is monotone and bounded, and its limit is 1.

Proof. For any $n \in \mathbb{N}$, $\frac{n+1}{n} = 1 + \frac{1}{n} > 1 + \frac{1}{n+1} = \frac{(n+1)+1}{n+1}$, so $\left(\frac{n+1}{n}\right)$ is monotonic and (strictly) decreasing. Therefore, it converges to $\inf_{n\in\mathbb{N}} \left\{\frac{n+1}{n}\right\}$. We claim $\inf \left\{\frac{n+1}{n}\right\} = 1$. Clearly, for all $n \in \mathbb{N}$, $1 < 1 + \frac{1}{n}$; furthermore, as a consequence of the Archimedean Property of the real numbers, for any $\epsilon > 0$, $1 + \frac{1}{n} < 1 + \epsilon$ for some n, so 1 is the greatest lower bound. $\square$

Remark. A similar argument shows that $\left(\frac{n-1}{n}\right) \to 1$, a fact which is useful for the next problem.

2.2.5 $\frac{n-\cos(n)}{n} \to 1$.

Proof. Since $-1 \le \cos(n) \le 1$, $\frac{n-1}{n} < \frac{n-\cos(n)}{n} < \frac{n+1}{n}$. Thus $\frac{n-\cos(n)}{n} \to 1$ by the Squeeze Lemma. $\square$

2.2.10 *Proof.* Let $x = \lim_{n\to\infty} x_n$, and let $\epsilon > 0$.

Case 1: $x = 0$. Then $\left|x_n^{\frac{1}{k}} - x\right| = \left|x_n^{\frac{1}{k}}\right|$. For a sufficiently large $N \in \mathbb{N}$, $n \ge N \Rightarrow |x_n| < \epsilon^k$; hence, $\left|x_n^{\frac{1}{k}}\right| < \epsilon$.

That was the easy case. For $x > 0$ we have to work harder, but once we see the method the calculation is easy. Notice that, in general, $(a^{\frac{1}{k}} - b^{\frac{1}{k}})(a^{\frac{k-1}{k}} + a^{\frac{k-2}{k}} b^{\frac{1}{k}} + a^{\frac{k-3}{k}} b^{\frac{2}{k}} + \cdots + b^{\frac{k-1}{k}}) = a - b$. So

$$x_n^{\frac{1}{k}} - x^{\frac{1}{k}} = \frac{x_n - x}{x_n^{\frac{k-1}{k}} + x_n^{\frac{k-2}{k}} x^{\frac{1}{k}} + x_n^{\frac{k-3}{k}} x^{\frac{2}{k}} + \cdots + x^{\frac{k-1}{k}}}.$$

Notice as well that all of the terms $x_n^{\frac{k-1}{k}}, x_n^{\frac{k-2}{k}} x^{\frac{1}{k}}, x_n^{\frac{k-3}{k}} x^{\frac{2}{k}}, \ldots, x_n^{\frac{1}{k}} x^{\frac{k-2}{k}}$ are positive. Thus,

$$\frac{|x_n - x|}{x_n^{\frac{k-1}{k}} + x_n^{\frac{k-2}{k}} x^{\frac{1}{k}} + x_n^{\frac{k-3}{k}} x^{\frac{2}{k}} + \cdots + x^{\frac{k-1}{k}}} < \frac{|x_n - x|}{x^{\frac{k-1}{k}}}.$$

Case 2. $x > 0$. For sufficiently large $N \in \mathbb{N}$, $n \geq N \Rightarrow |x_n - x| < \epsilon x^{\frac{k-1}{k}}$; hence $\left| x_n^{\frac{1}{k}} - x^{\frac{1}{k}} \right| < \epsilon$. (Now you can see why we had to separate out the case $x = 0$.

□

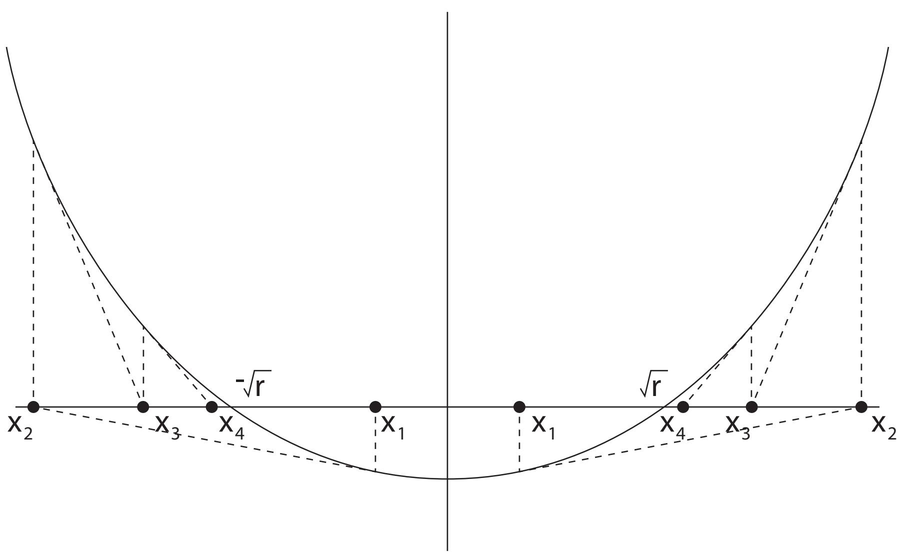
2.2.11 Let $r > 0$. The sequence recursively defined by

$$x_{n+1} = x_n - \frac{x_n^2 - r}{2x_n}$$

converges to $\sqrt{r}$, if $x_1 > 0$, and converges to $-\sqrt{r}$, if $x_1 < 0$.

Proof. First we must show the sequence converges. Then we can use the recursive definition to find the limit.

This is a tricky one! As I worked on it, I realized that this is the sequence given by Newton's method for solving $x^2 - r = 0$. The figure below illustrates both cases. (Check this for yourself; the figure should be sufficient to remind you how to do Newton's method.) Since $f(x) = x^2 - r$ is concave upward, we see that for $n \geq 2$, (x_n) is decreasing and bounded below by $\sqrt{r}$ if $x_1 > 0$ and (x_n) is increasing and bounded above by $-\sqrt{r}$ if $x_1 < 0$.



Since the tail of sequence from $n = 2$ on is monotonic and bounded in either case, the sequence converges. Denote its limit by x. Note also that $x > 0$ if $x_1 > 0$, and $x < 0$ if $x_1 < 0$ (why?). Using the recursive relationship, we obtain

$$x = x - \frac{x^2 - r}{2x} = \frac{x^2 + r}{2x} \Leftrightarrow 2x^2 = x^2 + r \Leftrightarrow x^2 - r = 0.$$

□

Remark. This is a significant result, because it proves that Newton's method actually works for this function, no matter what $x_1 \neq 0$ is chosen!

Remark. This is also the sequence given by the Babylonian algorithm, devised by the ancient Babylonians to estimate square roots, which calculates x_{n+1} by averaging x_n and $\frac{r}{n}$. (Why does this make sense? There is an elementary explanation.)

2.2.13 (a) If there exists $r < 1$ and $N \in \mathbb{N}$ such that, for all $n \geq N$, $\frac{|x_{n+1}|}{|x_n|} \leq r$, then $x_n \to 0$.

Proof. We easily prove by induction that $0 \leq |x_{N+k}| \leq |x_N| r^k$, for $k = 1, 2, 3, \ldots$. (Do it!) Thus, the tail of the sequence, and hence the sequence itself, converges by the Squeeze Lemma. (Here we made use of the Geometric Sequence Theorem, Prop. 2.2.11.) □

Part (b) is similar and left to you.

2.2.15 $\lim_{n\to\infty} \left(n^2+1\right)^{\frac{1}{n}} = 1.$

Proof. We will show that, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left(n^2+1\right)^{\frac{1}{n}} < 1+\epsilon$ for $n > N$. Since clearly $1 < \left(n^2+1\right)^{\frac{1}{n}}$ for all $n \in \mathbb{N}$, the result follows by definition of limit.

Now, $\left(n^2+1\right)^{\frac{1}{n}} < 1+\epsilon \Leftrightarrow \left(n^2+1\right) < (1+\epsilon)^n \Leftrightarrow \frac{(n^2+1)}{(1+\epsilon)^n} < 1$. We will show that $\lim_{n\to\infty} \frac{(n^2+1)}{(1+\epsilon)^n} = 0$; hence, there must be an $N \in \mathbb{N}$ such that $\frac{(n^2+1)}{(1+\epsilon)^n} < 1$ if $n > N$ (by definition of limit). The desired result follows from the ratio test:

$$\frac{\left((n+1)^2+1\right)}{(1+\epsilon)^{n+1}} \frac{(1+\epsilon)^n}{(n^2+1)} \to \frac{1}{1+\epsilon} < 1.$$

□

Remark. To show $\frac{\left((n+1)^2+1\right)}{(n^2+1)} \to 1$, distribute $(n+1)^2$, divide numerator and denominator by n^2, and use the algebraic properties of limits (Prop. 2.2.5).

Here is anotherway to write the proof of the general Bolzano-Weierstrass Theorem. It is a proof by induction. Writing this induction in a more formal way avoids the proliferation of subscripts, yielding a nicer presentation. To whit: let $(x_n) = ((x_{1,n}, x_{2,n}, x_{3,n}, \ldots x_{k,n}))$ be a bounded sequence in $\mathbb{R}^k$. We must show that (x_n) has a convergent subsequence. (Note that a sequence is convergent if and only each coordinate sequence is convergent. Why?) The case $k = 1$ has been proven. Proceeding by induction, assume the theorem holds for k; that is, any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence. Let $(x_n) = ((x_{1,n}, x_{2,n}, x_{3,n}, \ldots, x_{k,n}, x_{k+1,n}))$ be a bounded sequence in $\mathbb{R}^{k+1}$. Since $|\left((x_{1,n}, x_{2,n}, x_{3,n}, \ldots x_{k,n})\right)| \leq |\left((x_{1,n}, x_{2,n}, x_{3,n}, \ldots, x_{k,n}, x_{k+1,n})\right)|$ (why?), the sequence $((x_{1,n}, x_{2,n}, x_{3,n}, \ldots, x_{k,n}))$ is also bounded; hence it has a convergent subsequence $((x_{1,n_i}, x_{2,n_i}, x_{3,n_i}, \ldots, x_{k,n_i}))$, by inductive hypothesis. Furthermore, the sequence (x_{k+1,n_i}) is bounded, and therefore it has a convergent subsequence $\left(x_{k+1,n_{i_j}}\right)$. Thus, the subsequence $\left(x_{n_{i_j}}\right) = \left((x_{1,n_{i_j}}, x_{2,n_{i_j}}, x_{3,n_{i_j}}, \ldots, x_{k,n_{i_j}}, x_{k+1,n_{i_j}})\right)$ of (x_n) is convergent. (Note that we used the fact that any subsequence of a convergent sequence is convergent.)