MAT 4860: Selected Solutions to Problems

0.3.12 *If* $|A| = n$, *then* $|\mathcal{P}(A)| = 2^n$.

PROOF. If $n = 0$, $A = \emptyset$, so $\mathcal{P}(A) = \{\emptyset\}$; $|\{\emptyset\}| = 1 = 2^0$. For the general case, suppose as inductive hypothesis that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. Consider A with $|A| = n + 1$. Let a_0 be any fixed element of A. Then $|A \setminus \{a_0\}| = n$; hence, by inductive hypothesis, $|\mathcal{P}(A \setminus \{a_0\})| = 2^n$. Thus, there is a bijection $f : \{1, 2, 3, \ldots 2^n\} \to \mathcal{P}(A \setminus \{a_0\})$.

We construct a bijection from

$$\{1, 2, 3, \ldots 2^{n+1}\} = \{1, 2, 3, \ldots, 2^n\} \cup \{2^n + 1, 2^n + 2, 2^n + 3, \ldots, 2^n + 2^n = 2^{n+1}\}$$

to $\mathcal{P}(A)$ as follows. Note that, for every subset $B \subseteq A$, either $B \subseteq A \setminus \{a_0\}$, or $a_0 \in B$ and $B \setminus \{a_0\} \subseteq A \setminus \{a_0\}$. Furthermore, if $a_0 \in B$ and $a_0 \in B'$, then $B = B' \Leftrightarrow B \setminus \{a_0\} = B' \setminus \{a_0\}$. Define $g : \{1, 2, 3, \ldots 2^{n+1}\} \to \mathcal{P}(A)$ by

$$g(i) = \begin{cases} f(i), \text{ if } 1 \le i \le 2^n \\ f(i - 2^n) \cup \{a_0\}, \text{ if } 2^n + 1 \le i \le 2^{n+1} \end{cases} .$$

□

0.3.24 (a) By hypothesis, $A \subseteq B$, and B is countably infinite. Suppose A is not empty and A is not finite; we claim in this case that A is countably infinite. (Obviously, A is infinite, by definition of "not finite"! We need only show A is countable.) We have a bijection $f : \mathbb{N} \to B$, which we may represent by setting $b_n = f(n)$. Let n_1 be the smallest element of $\mathbb{N}$ such that $b_{n_1} \in A$; n_1 must exist, since $A \neq \emptyset$ and $\mathbb{N}$ is well-ordered. Proceeding recursively, suppose $n_1, n_2, n_3, \ldots, n_k$ have already been defined, and define n_{k+1} to be the smallest element of $\mathbb{N}$ such that $n_{k+1} > n_k$ and $n_k \in A$. Such an element must exist, since if it didn't, we would have $A \subseteq \{b_1, b_2, b_3, \ldots, b_{n_k}\}$, and we proved that a subset of a finite set is finite.

We thus obtain n_i for all $i \in \mathbb{N}$. In other words, we have recursively defined an injection $g : \mathbb{N} \to \mathbb{N}$, where $g(i) = n_i$. Let $a_i = b_{n_i}$. In other words, we obtain an injection $f \circ g : \mathbb{N} \to A$, where $a_i = f \circ g(i)$. We claim that $f \circ g$ is also a surjection onto A. For suppose by way of contradiction that there is some $a \in A$ such that $\forall i \in \mathbb{N}$, $a \neq b_{n_i}$. Since $A \subseteq B$, we know $a = b_n$ for some $n \in \mathbb{N}$. Setting j to be the smallest element of $\mathbb{N}$ such that $n < n_j$ (why must such a number exist?), we would have $n_{j-1} < n < n_j$, contradicting our recursive definition of n_j.

1.1.6 Assume S is an ordered set, $\emptyset \neq A \subseteq S$, A is bounded above, and $\sup A$ exists. Assume in addition that $\sup A \notin A$. Let a_1 be any element of A. Since a_1 cannot be equal to $\sup A$, it follows that $a_1 < \sup A$; therefore, a_1 is not an upper bound for A. More generally, no element of A can be an upper bound for A. Let a_2 be an element of A such that $a_1 < a_2$. Proceeding recursively, assume that $a_1 < a_2 < a_3 < \cdots < a_k$ have been chosen, and let a_{k+1} be an element of A such that $a_k < a_{k+1}$. In this manner we obtain a countably infinite subset $\{a_1, a_2, a_3, \ldots\}$. Note that our recursive definition requires the Axiom of Choice, because we have to have a chosen element of A at each stage "ready to go all at once” in order for the function $i \to a_i$ to be defined, and there is no rule by which to make this choice. We can achieve this by using the Axiom of Choice to select an element from every subset of A; then we will have one for each set $\{a \in A : a_i < a\}$. (If S were well-ordered, we could choose the smallest element in each subset of S, using this rule to avoid the Axiom of Choice, but S was not given to be well-ordered.) For a thorough and rigorous discussion of the Axiom of Choice, the Principle of Recursive Definition, and a variety of other foundational matters, see the introductory chapter of *Topology: a first course*, by J. Munkres.

1.1.8 It is clear how to define addition and multiplication, and it is clear that a finite field cannot be ordered (as a field); taking the field of three elements as a representative example, we would have $0 < 1 < 1+1 = 2 < 2+1 = 0$, violating anti-symmetry. I just want to make sure you understand why the addition and multiplication tables you created are the *only* way these operations may be defined on $\{0, 1, 2\}$. The key idea is that, thanks to the existence of inverses, $x+y = x+z \Rightarrow y = z$, and for any $x \neq 0$, $xy = xz \Rightarrow y = z$. Thus, each element may appear only once, and hence exactly once, in each row of the addition table. We know $1 + 0 = 1$ by definition of 0, so our choice is $1 + 1 = 0$ or $1 + 1 = 2$. But if $1 + 1 = 0$, we would have to have $1 + 2 = 2$, which would in turn imply that $1 = 0$, contradicting the implicit assumption that $1 \neq 0$. Now by process of elimination, since $2 + 0 = 2$ and $2 + 1 = 0$, we must have $2 + 2 = 1$. For the multiplication table, we proved that $\forall x \in \mathbb{F}, x \cdot 0 = 0$, and the definition of 1 implies that $\forall x \in \mathbb{F}, x \cdot 1 = x$. Thus, the only entry left in the multiplication table is $2 \cdot 2$. We proved that, in any field, $xy = 0 \Leftrightarrow x = 0$ or $y = 0$ (make sure you know how to prove that!), so $2 \cdot 2 \neq 0$. And $2 \cdot 1 = 2$, so by process of elimination we must have $2 \times 2 = 1$.

1.1.12 In any ordered field $\mathbb{F}$, $1 = 1^2 > 0$; hence, $0 < 1 < 1+1 < 1+1+1 < \cdots$. To write this rigorously, recursively define an injection $\mathbb{N} \to \mathbb{F}$ in this way and identify $n \in \mathbb{N}$ with its image in $\mathbb{F}$. (In other words, identify $n+1 \in \mathbb{N}$ with the sum of $n \in \mathbb{F}$, as already defined, and $1 \in \mathbb{F}$. There is a “copy” of the natural numbers, and more generally the integers, in any ordered field, and it is natural, if a bit confusing, to use the same notation in both contexts.)