Analysis Practice Problems

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If $f : D \to E$ is continuous at x and $g : E \to F$ is continuous at $y = f(x)$, then $g \circ f$ is continuous at x. (Here, D, E, and F are sets of real numbers.)

If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are continuous at $x \in D$, then so is fg.

If f is continuous on a closed and bounded interval, then f is uniformly continuous on that interval.

Prove or give a counter-example for each of the following:

If f is Lipschitz continuous, then f is uniformly continuous.

If f is uniformly continuous, then f is Lipschitz continuous.

3.4.14

Suppose $a \leq a_n < b_n \leq b$, $a_n \to a$, $b_n \to b$.

Prove or give a counter-example for each of the following:

If f is continuous on $[a_n, b_n]$ for all n, then f is continuous on $[a, b]$.

If f is continuous on $[a_n, b_n]$ for all n, then f is continuous on (a, b).

State and prove the product rule for differentiation.

State and prove the quotient rule for differentiation.

State and prove the chain rule for differentiation.

4.1.15

State and prove the Mean Value Theorem.

State and prove Taylor's Theorem.

Define Riemann integrability and the Riemann integral, giving all preliminary definitions and stating and proving all preliminary results.

State and prove additivity of the Riemann integral.

State and prove linearity of the Riemann integral.

State and prove monotonicity of the Riemann integral.

5.2.14

If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.

If $f = g$ on $[a, b]$ except at finitely many points, and $g \in \mathcal{R}[a, b]$, then $f \in \mathcal{R}[a, b]$, and $\int_a^b f = \int_a^b g$. (Hint: The proof requires a preliminary lemma involving $a \leq a_n < b_n \leq b$, $a_n \to a$, $b_n \to b$.)

State and prove the Fundamental Theorem of Calculus, Part 1. (This is the part for which a continuous function $F : [a, b] \to \mathbb{R}$ that is differentiable on (a, b) is part of the hypothesis.)

State and prove the Fundamental Theorem of Calculus, Part 2

5.3.8

Suppose $f_n : D \to R$ are continuous for all n.

Prove or give a counter-example for each of the following:

If f_n converges to a function f, then f is continuous on D.

If f_n converges uniformly to a function f, then f is continuous on D.

Suppose $f_n : D \to R$ are bounded for all n. $\{f_n\}$ is uniformly Cauchy if and only if $f_n \to f$ uniformly for some function $f : D \to \mathbb{R}$.

Suppose $f_n : [a, b] \to R$ are Riemann integrable for all n.

Prove or give a counterexample for each of the following:

If $f_n \to f$, then $f \in \mathcal{R}[a, b]$, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

If $f_n \to f$ uniformly, then $f \in \mathcal{R}[a, b]$, and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

State and prove conditions under which a sequence of differentiable functions $\{f_n\}$ converges to a differentiable function f such that $f' = \lim n \to \infty f_n'$. Make these conditions and weak as you can!

6.2.11