Analysis Practice Problems

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October 12, 2019

Define $|A| = |B|$, $|A| \leq |B|$, and $|A| < |B|$, where $|\ |$ denotes cardinality.

For any set A, $|A| < |\mathcal{P}(A)|$.

The Archimedean Property: for any $x \in \mathbb{R}$, there is a natural number $n \in \mathbb{N}$ such that $x < n$.

Consequences of the Archimedean Property:

For any $x \in \mathbb{R}$ such that $x > 0$, there is a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

Density of the rationals in the reals: given any real numbers $x < y$, there is a rational number r such that $x < r < y$.

Let $\emptyset \neq S \subseteq \mathbb{Z}$. If S is bounded below, then S has a minimum element.

True or false: if $S \subseteq \mathbb{R}$ and $T \subseteq \mathbb{R}$ are bounded above, and $S + T = \{s + t : s \in S \text{ and } t \in T\}$, then $S + T$ is bounded above, and $\sup(S + T) = \sup S + \sup T$.

If true, prove it. If false, give a counterexample.

Let $S \subseteq \mathbb{R}$ be a bounded set with no maximum and no minimum. Then S contains both a strictly monotone increasing sequence and a strictly monotone decreasing sequence. Furthermore, these sequences can be chosen so that the former converges to $\sup S$ and the latter converges to $\inf S$.

If a sequence converges, then it is bounded.

A monotone sequence converges if and only if it is bounded. (Prove decreasing case.)

If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} x_n y_n = xy$.

$\lim_{n\to\infty} c^n = 0$ if $|c| < 1$; (c^n) diverges if $c > 1$. What happens if $c = \pm 1$?

State and prove the Ratio Test for Sequences.

Is there a Root Test for sequences? If so, state and prove it.

Let (x_n) be a bounded sequence. Let $a_n = \inf\{x_k : k \geq n\}$ and $b_n = \sup\{x_k : k \geq n\}$.

Define $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ in terms of the sequences (a_n) and (b_n) and show they exist.

In addition, prove that $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.

A sequence (x_n) converges if and only if it is bounded and $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, in which case

$$\lim_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Can the Ratio Test for Sequences be improved using $\limsup$ or $\liminf$? Can the Root Test?

State and prove the Bolzano-Weierstrass Theorem.

Define what it means for a sequence to be *Cauchy* and prove that a sequence converges if and only if it is Cauchy.